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SUMMARY REPORT

ON GENERAL PERTURBATION METHODS FOR CELESTIAL MECHANICS AND DIFFERENTIAL CORRECTION SCHEMES FOR THE CALCULUS OF VARIATIONS

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THE HAMILTON-JACOBI FORMULATION OF THE RESTRICTED THREE BODY PROBLEM IN TERMS OF THE TWO FIXED CENTER PROBLEM

By

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Summary

This report contains a development of the classical Hamilton-Jacobi perturbation techniques, applying the known solution of the Two Fixed Center Problem to the Restricted Three Body Problem.

SECTION I - INTRODUCTION

This report contains an outline of the development of a perturbation procedure for solving the restricted three body problem, using the solution of the two fixed center problem as an intermediate orbit. In the restricted problem, it is assumed that the two primary bodies move in circles about their center of mass, the barycenter. The primary bodies will be fixed in a coordinate system rotating with their angular velocity, so that the use of the two fixed center problem is immediately suggested. The two fixed center problem was first treated by Euler, who discovered that its equations of motion are separable in prolate spheriodial coordinates. A very complete discussion of the two fixed center problem has been given by Charlier[(1)]. This treatment covers some of the same ground as this report. It is from the

Hamiltonian point of view and includes a discussion of the action and angle variables, and the way in which the two fixed center problem would be used as a basis for a perturbation theory for the restricted problem. The only thing missing from Charlier's treatment is an explicit solution of the two fixed center problem, which would be necessary for the actual application to the restricted problem. Formal expressions for the action and angle variables are obtained from a more modern point of view by Buchheim[(2)]. Brief discussions of the two fixed center problem are given in many standard text books such as Whittaker[(3)], Landau and Lifschitz[(4)] and Wintner[(5)]. The explicit solution of the two fixed center problem has been obtained by Pines and Payne[(6)]. In the present report, this solution will be combined with a Hamiltonian development of the problem to show how perturbation equations for the restricted problem may be obtained. A different development has been carried out recently by Davidson and Schulz-Arenstorff[(7)]. In this theory, the initial conditions of a two fixed center problem are used as parameters and a first order correction for the restricted problem is obtained in closed form. Second-order corrections are obtained by a numerical curve-fitting scheme.

In this report, Section II will contain a discussion of the restricted problem, and the way in which the two fixed center problem will be used. In Section III, the solution of the two fixed center problem will be outlined in sufficient detail for the determination of the action and angle variables, which is carried out in Sections IV and V. Finally in Section VI a summary will be given of the essential steps still necessary to obtain the solution of the restricted problem.

SECTION II - THE RESTRICTED PROBLEM

The equations of motion of the restricted problem are

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \tag{1}$$

where $\underline{R}$ is the position vector of the vehicle in a coordinate system fixed in space, $\underline{R}_1$ and $\underline{R}_2$ are respectively the position vectors of the vehicle from earth and moon (with magnitudes r_1 and r_2), and μ and μ' are the gravitational constant times mass of the earth and moon, respectively. Since the barycenter (center of mass of earth and moon) may be regarded as a point fixed in space, the vector $\underline{R}$ will henceforth be regarded as relative to a system fixed in space with origin at the barycenter. The earth and moon are taken as moving in circles about the barycenter with angular velocity vector $\underline{\Omega}$. To use the two fixed center problem as an approximation to the restricted problem, it is necessary to write the equations of motion in a coordinate system in which the earth and moon are fixed. Such a system is one rotating about the barycenter with angular velocity $\underline{\Omega}$ relative to the fixed system. Denoting the position vector in the rotating system by $\underline{R}'$, the equations of motion (1) become

$$\ddot{\underline{R}}' = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} - 2\,\underline{\Omega} \times \dot{\underline{R}}' - \underline{\Omega} \times (\underline{\Omega} \times \underline{R}') \tag{2}$$

It is readily shown that the Lagrangian for the equations of motion (2) is

$$\mathcal{L} = \frac{1}{2}\, \dot{R}'^2 + \underline{\Omega} \cdot \underline{R}' \times \dot{\underline{R}}' + \frac{1}{2}\, (\underline{\Omega} \times \underline{R}')^2 + \frac{\mu}{r_1} + \frac{\mu'}{r_2} \tag{3}$$

and hence the momentum vector conjugate to the position vector $\underline{R}'$ is given by

$$\underline{P} = \text{grad}_{\dot{R}'} \mathcal{L} = \dot{\underline{R}}' + \underline{\Omega} \times \underline{R} \tag{4}$$

and the Hamiltonian for the problem is

$$H = \underline{P} \cdot \dot{\underline{R}}' - \mathcal{L} = \frac{1}{2} P^2 - \underline{\Omega} \cdot \underline{R}' \times \underline{P} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \tag{5}$$

and the Hamiltonian equations are

$$\dot{\underline{P}} = -\text{grad}_{R'} H = -\underline{\Omega} \times \underline{P} - \mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \tag{6}$$

and

$$\dot{\underline{R}}' = \text{grad}_P \; H = \underline{P} - \underline{\Omega} \times \underline{R}' \tag{7}$$

It will be noted that Eq. (7) is equivalent to Eq. (4), and that if $\underline{P}$ is replaced using Eq. (4), then Eq. (6) will yield the equations of motion (2).

The solution of the restricted problem will be carried out by making use of a transformation theorem (Reference 1, Chapter 11 and Reference 12, pp 237 to 246) which states that if the Hamiltonian of a system is $H(q_i, p_i, t)$ with q_i and p_i canonically conjugate coordinates so that the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{8}$$

are satisfied and if $\psi(q_i, P_i, t)$ is any function, then the variables Q_i and P_i defined by

$$Q_i = \frac{\partial \psi}{\partial P_i} = Q_i(q_i, P_i, t), \qquad p_i = \frac{\partial \psi}{\partial q_i} = p_i(q_i, P_i, t) \tag{9}$$

are canonical variables for a new Hamiltonian

$$\overline{H} = H + \frac{\partial \psi}{\partial t} \tag{10}$$

regarded as a function of Q_i, P_i and t, so that

$$\dot{Q}_i = \frac{\partial \overline{H}}{\partial P_i} \quad , \qquad \dot{P}_i = - \frac{\partial \overline{H}}{\partial Q_i} \tag{11}$$

Now let the Hamiltonian be separated into two terms

$$H = H_1 (q_i, p_i) + H_2 (q_i, p_i, t) \tag{12}$$

with H_1 independent of the time and such that the partial differential equation

$$H_1 (q_i, \frac{\partial \psi_1}{\partial q_i}) + \frac{\partial \psi_1}{\partial t} = 0 \tag{13}$$

possesses a solution for ψ_1. It is seen that if the function ψ_1 is used in the transformation theorem then the Hamilton equations become

$$\begin{aligned} \dot{Q}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial \psi_1}{dt})}{\partial P_i} = \frac{\partial H_2}{\partial P_i} \\ \dot{P}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial \psi_1}{\partial t})}{\partial Q_i} = - \frac{\partial H_2}{\partial Q_i} \end{aligned} \tag{14}$$

by virtue of the defining Eq. (13) for ψ_1. Further, from Eq. (13), it is evident that, since H_1 is independent of time,

$$\psi_1 = - ht + W(q_i \, , \, P_i) \tag{15}$$

with

$$H_1 (q_i, \frac{\partial W}{\partial q_i}) - h = 0 \tag{16}$$

and the momenta P_i must be identified with the constants of integration of Eq. (16) and h, the separation constant for the time. This is not to be interpreted as meaning that the P_i are constants of the motion for the Hamiltonian H. If this were so, the right-hand sides of Eq. (14) would have to vanish. What the solution of Eq. (16) for W does is to specify a function of q_i and three new variables P_i.

This function may be used to invert Eqs. (9) to obtain q_i and p_i in terms of the new variables P_i and three others Q_i. These expressions for q_i and p_i may now be inserted in H_2 for use in Eqs. (14) from which Q_i and P_i may now be obtained as functions of time. The solution of the problem associated with H will then be given by substituting the solutions Q_i and P_i of Eqs. (14) in the expression for q_i and p_i.

To actually carry out the inversion of Eqs. (9) it must be noted that the functional form of ψ_1 does not depend on the disturbing function ultimately to be used. It depends rather on how the identification of the P_i is made with the integration constants arising in Eq. (16). The conventional procedure is to regard H_1 as the Hamiltonian of a new problem and identify the P_i with the action variables J_i of this new problem. The action variables are always three independent functions of the integration constants and hence are themselves constant for the problem associated with H_1. Once the functional relation between the P_i and the integration constants is determined, by identifying the P_i with the action variables J_i of H_1, the conjugate coordinates Q_i are defined by Eq. (9). It will always happen that P_i and Q_i so defined are constant if the Hamiltonian is H_1 because from Eq. (13)

$$\dot{Q}_i = \frac{\partial \left(H_1 + \frac{\partial \psi_1}{\partial t}\right)}{\partial P_i} = 0$$

$$J_i = \dot{P}_i = - \frac{\partial \left(H_1 + \frac{\partial \psi_1}{\partial t}\right)}{\partial Q_i} = 0 \tag{17}$$

Once the functional relation between q_i and p_i and Q_i and J_i is established, however, the problem associated with H_1 is no longer of interest. The disturbing function H_2 is expressed in terms of Q_i and J_i and the solution of the problem associated with H is obtained by integrating Eqs. (14).

A slightly different formulation of the problem is obtained if the time independent function W of Eq. (15) is used as the generating function of the transformation rather than ψ_1. The variables w_i conjugate to the action variables

J_i are the angle variables of the problem associated with H_1. The relations between the w_i and the Q_i are given by

$$w_i = \frac{\partial W}{\partial J_i} = \frac{\partial(\psi_1 + ht)}{\partial J_i} = Q_i + t\frac{\partial h}{\partial J_i} = \nu_i t + Q_i \tag{18}$$

with

$$\nu_i = + \frac{\partial h}{\partial J_i} \tag{19}$$

being functions of the action variables. The perturbation equations for these variables will be given, according to the transformation theorem, by

$$\begin{aligned} \dot{w}_i &= \frac{\partial\left(H_1 + H_2 + \frac{\partial W}{\partial t}\right)}{\partial J_i} = \frac{\partial H_2}{\partial J_i} + \nu_1 \\ \dot{J}_i &= -\frac{\partial\left(H_1 + H_2 + \frac{\partial W}{\partial t}\right)}{\partial w_i} = -\frac{\partial H_2}{\partial w_i} \end{aligned} \tag{20}$$

since W is independent of time and $H_1 = h$ depends only on the action variables and not on the angle variables. The advantage of using the angle variables rather than the Q_i is that it will always be possible to expand H_2 in a multiple Fourier series in the angle variables and eliminate its explicit dependence on time.

To use the two fixed center problem to solve the restricted problem, the Hamiltonian (5) for the restricted problem may be separated into terms H_1 and H_2 as follows:

$$H_1 = \frac{1}{2} p^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \tag{21}$$

$$H_2 = -\underline{\Omega} \cdot \underline{R}' \times \underline{P} \tag{22}$$

If H_1 is regarded as a Hamiltonian, the associated Hamilton equations are

$$\dot{\underline{R}}' = -\operatorname{grad}_P H_1 = \underline{P} \tag{23}$$

and

$$\dot{\underline{P}} = - \text{grad}_{R'} H_1 = - \mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} = \ddot{\underline{R}}' \tag{24}$$

These last equations are just the equations of motion for the two fixed center problem, so that H_1 is the Hamiltonian of the two fixed center problem. Thus the procedure will be first to find the action and angle variables of the two fixed center problem and then express the disturbing function

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \times \underline{P} \tag{25}$$

in terms of these variables.

Before proceeding with the details of this procedure, it is desirable to make two further transformation of the coordinates. The first will be to a coordinate system with the origin at the midpoint of the earth-moon line with the earth and moon on the x-axis at (c,0,0) and (-c,0,0) respectively. The distance between the earth and moon is thus 2c. The z-axis will be taken in the direction of $\underline{\Omega}$. The only term in the Hamiltonian affected by this transformation is the $\underline{\Omega} \cdot \underline{R}' \times \underline{P}$ term in which $\underline{R}'$ is measured from the barycenter. From Figure I it is evident

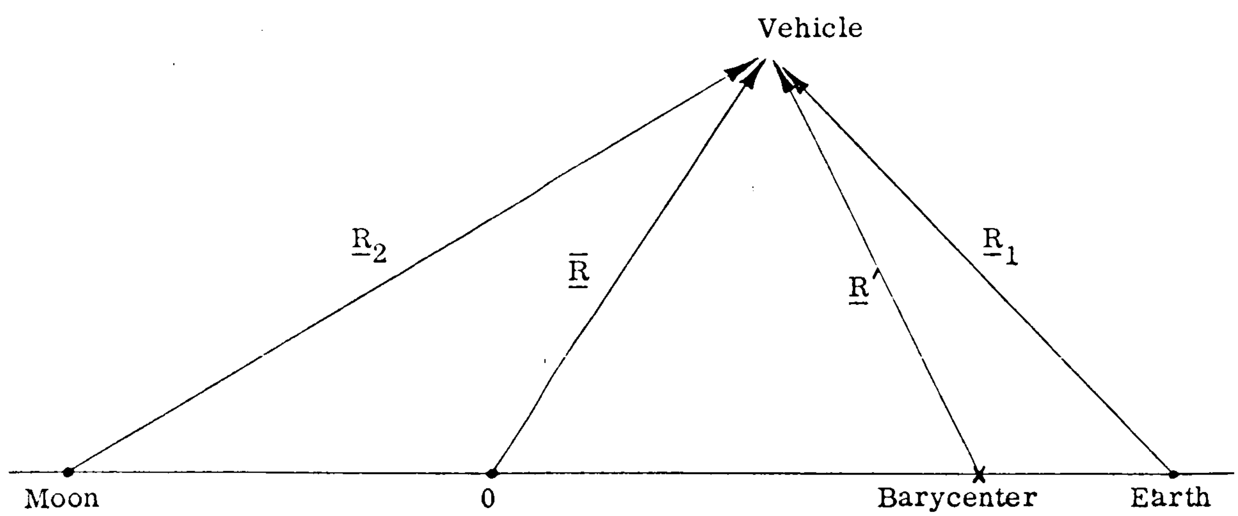


Figure I

From the equations for x, y, and z it is seen that

$$
\begin{aligned}
x &= c q_1 q_2 \\
y &= c\sqrt{(q_1^2 - 1)(1 - q_2^2)} \cos\varphi \qquad (30) \\
z &= c\sqrt{(q_1^2 - 1)(1 - q_2^2)} \sin\varphi
\end{aligned}
$$

In this system the surfaces $q_1 = \text{const} \geq 1$ are ellipsoids of revolution about the x-axis confocal about the earth and moon. The limiting surface $q_1 = 1$ is the portion of the x-axis between the earth and moon, and the ellipsoids increase in size with increasing q_1. The surfaces $-1 \leq q_2 = \text{const} \leq 1$ are hyperboloids of revolution about the x-axis, confocal about the earth and moon. The limiting surfaces $q_1 = 1$ and $q_2 = -1$ are the portions of the x-axis to the right of the earth and to the left of the moon, respectively. The surface $q_2 = 0$ is the y-z plane and surfaces corresponding to positive values of q_2 are hyperboloids concave towards the earth while those corresponding to negative values of q_2 are concave towards the moon. The angle φ is measured in the y-z plane about the x-axis and is zero in the portion of the x-y plane for which $y > 0$. From Eq. (30), it is easy to show that r_1 and r_2 which appear in the Hamiltonian (5) are given by

$$
\begin{aligned}
r_1 &= c(q_1 - q_2) \\
r_2 &= c(q_1 + q_2)
\end{aligned} \qquad (31)
$$

The equations for p_1, p_2, p_φ are

$$
\begin{aligned}
p_1 &= c q_2 P_x + \frac{c q_1 (1 - q_2^2) \cos\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_y + \frac{c q_1 (1 - q_2^2) \sin\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_z \\
p_2 &= c q_1 P_x - \frac{c q_2 (q_1^2 - 1) \cos\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_y - \frac{c q_2 (q_1^2 - 1) \sin\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_z
\end{aligned} \qquad (32)
$$

$$p_\varphi = -c\sqrt{(q_1^2-1)(1-q_2^2)}\,\sin\varphi\, P_y + c\sqrt{(q_1^2-1)(1-q_2^2)}\,\cos\varphi\, P_z \tag{32}$$

Inverting these equations to obtain P_x, P_y and P_z in terms of p_1, p_2 and p_φ one obtains for H_1

$$\begin{aligned} H_1 &= \frac{1}{2}P^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \\ &= \frac{1}{2c^2}\left[\frac{(q_1^2-1)\,p_1^2}{q_1^2-q_2^2} + \frac{(1-q_2^2)\,p_2^2}{q_1^2-q_2^2} + \frac{p_\varphi^2}{(q_1^2-1)(1-q_2^2)}\right] \\ &\qquad - \frac{\mu}{c\,(q_1-q_2)} - \frac{\mu'}{c\,(q_1+q_2)} \end{aligned} \tag{33}$$

and for the disturbing function

$$\begin{aligned} H_2 = \omega \Bigg\{ &\frac{\sqrt{(q_1^2-1)(1-q_2^2)}}{q_1^2-q_2^2}\cos\varphi\left[p_1q_2 - p_2q_1 + \frac{\mu-\mu'}{\mu+\mu'}(p_1q_1 - p_2q_2)\right] \\ &- \frac{p_\varphi \sin\varphi}{\sqrt{(q_1^2-1)(1-q_2^2)}}\left(q_1q_2 + \frac{\mu-\mu'}{\mu+\mu'}\right)\Bigg\} \quad . \end{aligned} \tag{34}$$

This completes the preliminary discussion of the problem. The following sections contain the solution of the two fixed center problem which will be useful in the subsequent determination of the generating function W from Eq. (16) and the action and angle variables for the two fixed center problem which will be the w_i and J_i of the perturbation Eqs. (19).

SECTION III - SOLUTION OF THE TWO FIXED CENTER PROBLEM

The Hamiltonian for the two fixed center problem, obtained in the last section is

$$H = \frac{1}{2c^2}\left\{\frac{q_1^2-1}{q_1^2-q_2^2}\,p_1^2 + \frac{(1-q_2^2)}{q_1^2-q_2^2}\,p_2^2 + \frac{p_\varphi^2}{(q_1^2-1)(1-q_2^2)}\right\} - \frac{\mu}{c\,(q_1-q_2)} - \frac{\mu'}{c\,(q_1+q_2)} \tag{35}$$

The generating function $W(q_1, q_2, \varphi, P_1, P_2, P_3)$, which will ultimately be used to obtain the w_i and P_i for the perturbation equations is also a very convenient device for obtaining a direct solution to the two fixed center problem. Recalling that for the transformation to be canonical, one must have

$$\begin{aligned} p_1 &= \frac{\partial W}{\partial q_1} \\ p_2 &= \frac{\partial W}{\partial q_2} \\ p_\varphi &= \frac{\partial W}{\partial \varphi} \end{aligned} \tag{36}$$

and

$$Q_i = \frac{\partial W}{\partial P_i} \quad . \tag{37}$$

Replacement of p_1, p_2 and p_φ by the partials of W with respect to q_1, q_2, and φ, respectively, in Eq. (35) gives a partial differential equation for W which is separable. That is, a solution of the form

$$W = W_1\,(q_1, P_i) + W_2\,(q_2, P_i) + W_3\,(\varphi, P_i) \tag{38}$$

exists. It is a fairly simple matter to verify that

$$\left(\frac{dW_1}{dq_1}\right)^2 = \left(\frac{\partial W}{\partial q_1}\right)^2 = p_1^2 = \frac{2c^2}{(q_1^2 - 1)^2} R^2 (q_1)$$

$$\left(\frac{dW_2}{dq_2}\right)^2 = \left(\frac{\partial W}{\partial q_2}\right)^2 = p_2^2 = \frac{2c^2}{(1 - q_2^2)^2} S^2 (q_2) \tag{39}$$

$$\left(\frac{dW_3}{d\varphi}\right)^2 = \left(\frac{\partial W}{\partial \varphi}\right)^2 = p_\varphi^2 = \alpha^2$$

where

$$R^2 (q_1) = (q_1^2 - 1) (hq_1^2 + \frac{\mu + \mu'}{c} q_1 - \beta) - \frac{\alpha^2}{2c^2} \tag{40}$$

$$S^2 (q_2) = (1 - q_2^2) (-hq_2^2 + \frac{\mu - \mu'}{c} q_2 + \beta) - \frac{\alpha^2}{2c^2} \tag{41}$$

In Equations (40) and (41), h is the constant energy of the two fixed center problem and is to be identified with the constant h of Equation (15) in the previous section. The separation constants are α and β. It is easily shown that α is the x-component of angular momentum about the line of centers. The constant β has no such simple interpretation.

At this stage everything necessary for the solution of the two fixed center problem is available; further discussion of the generating function will be deferred to the next section.

The Hamilton equations for the two fixed center problem give the time derivatives of q_1, q_2 and φ as

$$\dot{q}_1 = \frac{\partial H_1}{\partial p_1} = \frac{p_1}{c^2} \frac{q_1^2 - 1}{q_1^2 - q_2^2}$$

$$\dot{q}_2 = \frac{\partial H_1}{\partial p_2} = \frac{p_2}{c^2} \frac{1 - q_2^2}{q_1^2 - q_2^2} \tag{42}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_{\varphi}} = \frac{p_{\varphi}}{c^2 (q_1{}^2 - 1)(1 - q_2{}^2)} \tag{42}$$

Combination of these equations with Equation (39) yields

$$\dot{q}_1 = \frac{\sqrt{2}}{c} \frac{R(q_1)}{q_1{}^2 - q_2{}^2}$$

$$\dot{q}_2 = \frac{\sqrt{2}}{c} \frac{S(q_2)}{q_1{}^2 - q_2{}^2} \tag{43}$$

$$\dot{\varphi} = \frac{\alpha}{c^2 (q_1{}^2 - 1)(1 - q_2{}^2)}$$

A preferable form for these equations is the following in which a parameter u is introduced which completes the separation of the variables:

$$\frac{dq_1}{R} = \frac{dq_2}{S} = du \tag{44}$$

$$dt = \frac{c}{\sqrt{2}} (q_1{}^2 - q_2{}^2)\, du \tag{45}$$

$$d\varphi = \frac{\alpha}{c\sqrt{2}} \left[\frac{1}{q_1{}^2 - 1} + \frac{1}{1 - q_2{}^2} \right] du \tag{46}$$

From Equation (44), which leads to elliptic integrals of the first kind, q_1 and q_2 turn out to be expressible as elliptic functions of u. Using these expressions for q_1 and q_2 in Equations (45) and (46), it is then possible to obtain t and φ as functions of u. The integration of Equations (45) and (46) involves elliptic integrals of the second and third kinds.

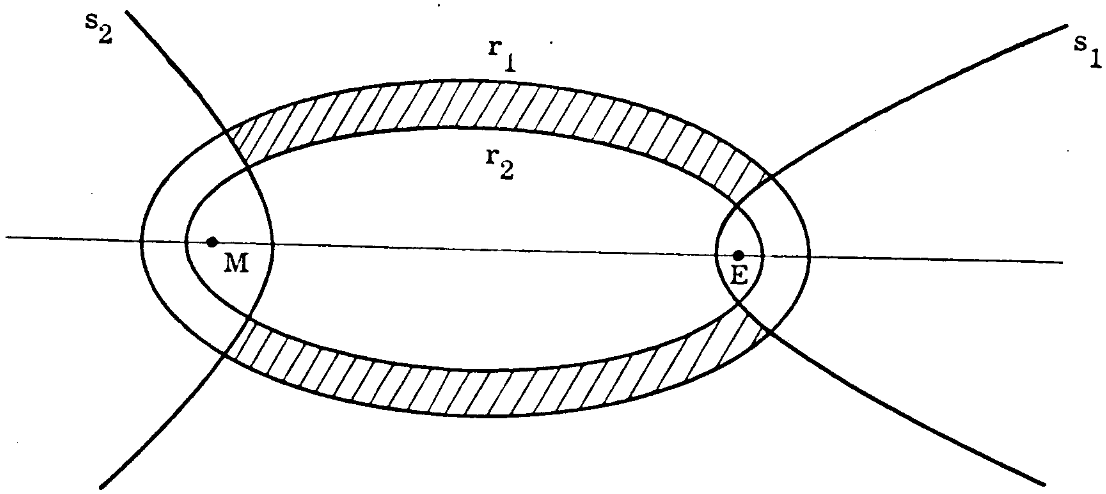
The form of solution of Equation (44) depends on the nature of the roots of the quartic expressions R^2 and S^2. These roots are uniquely determined by the three dynamical constants h, α and β. It is shown in Reference 6 that if $h < 0$, R^2 must have four real roots, two of which exceed unity and the other two lie in the interval (± 1). Further, R^2 is positive between the largest roots and also between the smallest. Since, however, q_1 must exceed unity, it follows that q_1 is constrained between the largest roots. Thus, if the roots of R^2 in order of decreasing magnitude are denoted by r_1, r_2, r_3, r_4 it may be said that

$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1 \tag{47}$$

This conclusion may be stated a little differently: the bounds r_1 and r_2 on q_1 represent two ellipsoids (the larger corresponding to r_1) which bound the region in space in which the vehicle may move.

The corresponding results for the quartic S^2 are more complicated: none of the roots exceed unity and at least two lie in the interval (± 1). The other two may also lie in this interval, may be real and both less than -1, or may be complex. The quartic is positive between the two largest roots and between the two smallest, if they are real. Since q_2 must lie in the interval (± 1) it follows that the orbit is constrained between the two largest roots or between the two smallest if they also lie in the (± 1) interval. If all four roots of S^2 are in (± 1), knowledge of the position of one point of the orbit specifies whether q_2 is constrained between the largest or the smallest roots; transitions from one band to the other cannot occur, since if S^2 becomes negative, $\dot{q}_2$ becomes imaginary. The roots of S^2 in the interval (± 1) correspond to hyperboloids bounding the motion in space.

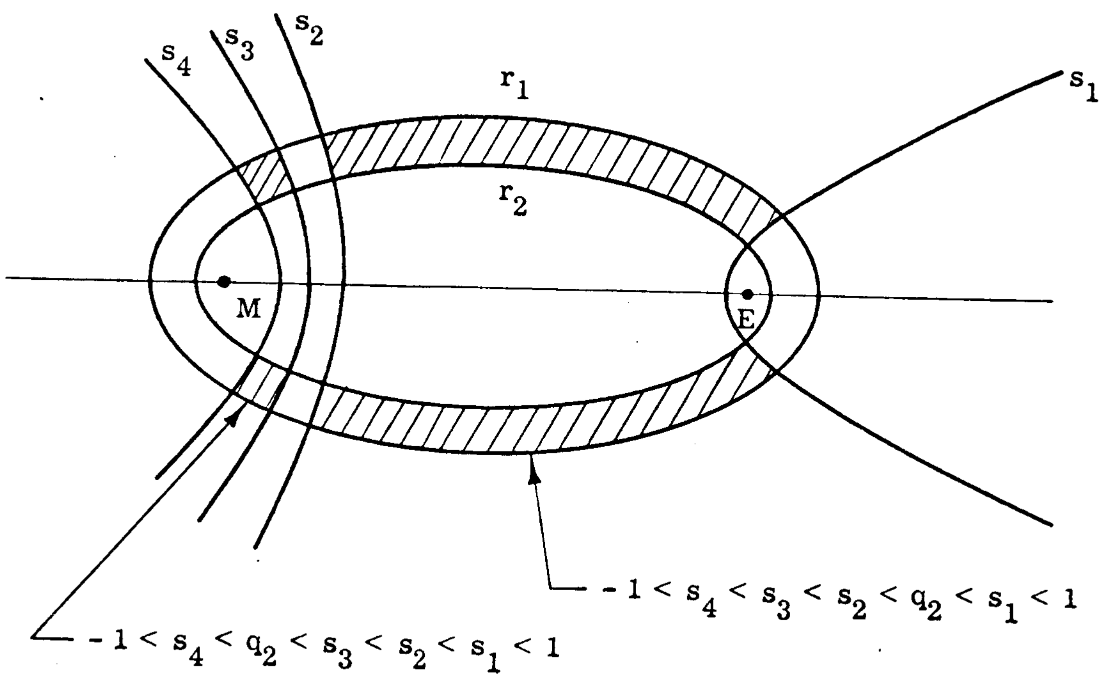
Summarizing the above results for negative energy, two possibilities for bounds on the orbit occur. These are shown in Figures II and III where the shaded areas are regions in which motion may occur.



$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1$$

$$-1 < s_2 < q_2 < s_1 < 1 \begin{cases} \text{either} & s_3, s_4 < -1 \\ \text{or} & s_3, s_4 \text{ complex} \end{cases}$$

Figure II



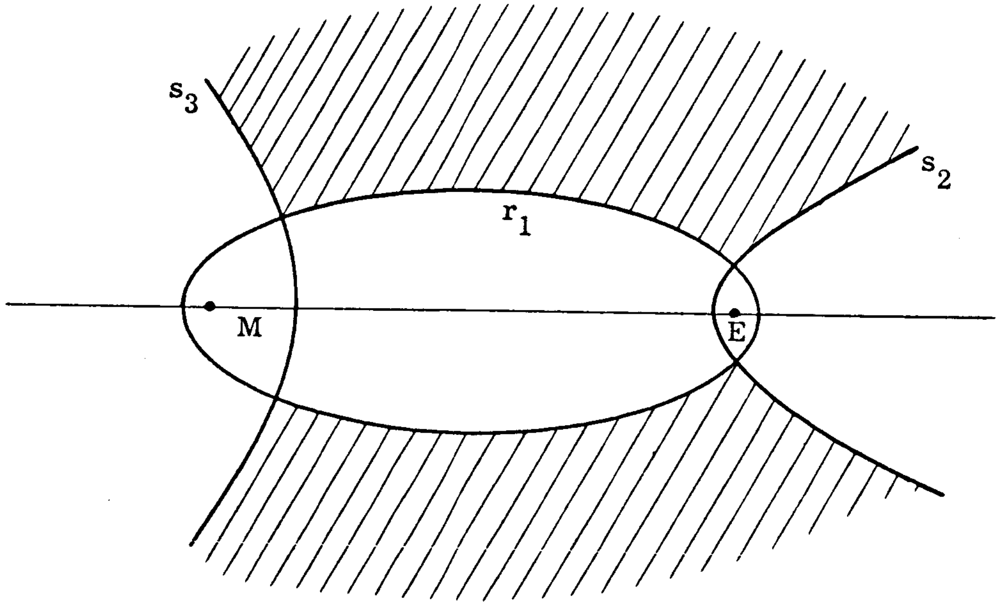
$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1$$

Figure III

If one thinks of h, α and β, which determine all the roots of R^2 and S^2 as being three dynamical specifications of a two fixed center orbit, it is clear that any remaining specifications must not violate the bounds on the region in which the motion can occur. That is, these bounds impose constraints on any further specifications. Actually, not even h, α and β can be arbitrarily selected: they must lead to roots of R^2 satisfying Equation (47) and roots of S^2 satisfying one or the other of the following:

$$
\begin{array}{ll}
\text{(a)} & -1 \le s_2 \le s_1 \le 1 \text{ and either } s_3, s_4 < 1 \text{ or } s_3, s_4 \text{ complex} \\
\text{(b)} & -1 \le s_4 \le s_3 \le s_2 \le s_1 \le 1
\end{array}
\tag{48}
$$

If the energy is positive, it may be shown that R^2 has one root, say r_1 exceeding unity and is positive for q_1 exceeding r_1. The other roots are all less than 1. The quartic S^2 has two roots $s_3 < s_2$ in the interval (±1), and one on each side of this interval. It is positive for $s_3 < q_2 < s_2$. Thus in this case the motion must take place in the unbounded region shown in Figure IV.



$q > r_1 > 1$

$-1 < s_3 < q_2 < s_2 < 1$

Figure IV

As noted above, q_1 and q_2 are expressible in terms of elliptic functions of u. The particular elliptic function occurring depends on the nature of the roots. In all cases, see Reference 6,

$$q_i = \frac{A_i f(\alpha_i (u + \beta_i)) + B_i}{C_i f(\alpha_i (u + \beta_i)) - 1} \tag{49}$$

The A_i, B_i, α_i are constants depending only on the roots and hence on h, α and β. The constants β_i depend on h, α and β as well as whatever additional specifications are made to select a particular orbit. For q_1, the function f is an sn or dn function according as h is negative or positive. For q_2, $h < o$, f is an sn or cn function according as all four or only two of the roots are real and if $h > o$, f is a dn function. It is evident, of course, that q_1 and q_2 are individually periodic in the variable u. The periods of q_1 and q_2 are, however, in general non-commensurable, so that the motion in space of the vehicle will, in general, be nonperiodic. The quarter periods of q_1 and q_2 are usually denoted by K_1 and K_2, respectively, and it may be shown that these quarter periods depend only on the roots of R^2 and S^2, respectively, and hence only on h, α and β. From the way in which the β_i occur in Equation (49), it is evident that they represent a phase. In fact, it is assumed in Equation (49) that $u = o$ corresponds to some point on the orbit, say the initial point, and the β_i represent the variation in u required to get from this point to one of the extreme values of q_i - that is, to a point of tangency with one of the bounding ellipsoids for q_1, and with one of the bounding hyperboloids for q_2.

The integration of the equations for time and φ leads in all cases to the following forms (consult Reference 6)

$$t = (n_1 - n_2) u + F_1(u) + F_2(u) \tag{50}$$

$$\varphi = (m_1 + m_2) u + G_1(u) + G_2(u) \tag{51}$$

where n_1 and m_1 are constants depending on the roots of R^2, and n_2 and m_2 depend on the roots of S^2. For negative h, the functions $F_1(u)$ and $G_1(u)$ are periodic functions of u with period $2K_1$, while $F_2(u)$ and $G_2(u)$ are periodic with period $2K_2$. For positive h, the functions F_i and G_i become logarithmic.

SECTION IV - DETERMINATION OF THE GENERATING FUNCTION

The differential equations for the generating function, Eqs. (39), may be written

$$\frac{d W_1}{d q_1} = \frac{\partial W}{\partial q_1} = \frac{\sqrt{2}\, c}{q_1^{\,2} - 1} \quad R$$

$$\frac{d W_2}{d q_2} = \frac{\partial W}{\partial q_2} = \frac{\sqrt{2}\, c}{1 - q_2^{\,2}} \quad S \qquad (52)$$

$$\frac{d W_3}{d \varphi} = \frac{\partial W}{\partial \varphi} = \alpha$$

These are ordinary differential equations, and integration again leads to elliptic integrals. Before carrying out the integration, however, some discussion of the limits on the integrals is necessary. It will be recalled that the generating function was to be a function of six variables.

$$W = W(q_1, q_2, \varphi, P_1, P_2, P_3) \qquad (53)$$

and the differential equations (52) give only three of the six partial derivatives of W. Now the dependence of W on q_1, q_2 and φ can be carried by the upper limits of the integrals resulting from Eqs. (52). These upper limits should be simply q_1, q_2, and φ, respectively. Recalling further that the momenta P_i are supposed to be constants, and noting that three independent constants h, α and β already are explicitly in Eq. (52), it is evident that these three constants or some three independent functions of them must be identified with P_i. It is convenient at present to identify h, α and β themselves with P_i and defer to a later stage in the development any more complicated identification. If this is done, it now becomes obvious that the lower limits on the integrals must be either functions of h, α and β or absolute constants. This is so first because W is a function only of q_1, q_2, φ

and the P_i, and, since the integrals will be functions of their limits, only these quantities and absolute constants may be included in the limits. Secondly, the upper limits have already been taken as q_1, q_2, and φ, and recalling that the partials of W with respect to q_1, q_2 and φ must be p_1, p_2 and p_φ, no further dependence of W on q_1, q_2 and φ can be allowed without modifying the p's from which the equations (52) for W were obtained in the first place. The only remaining problem, then, is to select lower limits which depend only on h, α and β. For the integral for W_1, the variable is q_1 which has bounds on its variation. The bounds depend on h, α and β, and since r_1 is a bound whether the energy is positive or negative, it is a satisfactory lower limit. For W_2 the bounds vary with the particular conditions of the problem. However, for orbits approaching both Earth and Moon, the bound s_2 always occurs, and will be selected as the lower limit. For W_3, the situation is a little different. The variable is φ, and reference to Eq. (43) shows that φ has the sign of α and is thus monotone. Hence any absolute constant is acceptable as a lower limit and 0 will be selected. The generating function may now be written:

$$W(q_1, q_2, \varphi, h, \alpha, \beta) = W_1(q_1, h, \alpha, \beta) + W_2(q_2, h, \alpha, \beta) + W_3(\varphi, h, \alpha, \beta)$$

$$= \sqrt{2}\, c \int_{r_1}^{q_1} \frac{R}{q_1^2 - 1}\, d q_1 + \sqrt{2}\, c \int_{s_2}^{q_2} \frac{S}{1 - q_2^2}\, d q_2 + \alpha \varphi \qquad (54)$$

where W_3 is integrable directly. It might be remarked at this stage that there is an essential difference between this generating function and the corresponding function for the Kepler problem. The upper limits in the integral occurring in both generating functions may be regarded as the coordinates of a point on the orbit. In the Kepler problem, the lower limits correspond to the perigee distance for the radial integral and to zero for the two angle integrals. This may be regarded as a point on any orbit, since the angles may just be measured from the perigee point. In the two fixed center problem however, the lower limits r_1, s_2 and 0 may be regarded as a point only on a very special orbit -- namely, one which is simultaneously tangent to the ellipsoid r_1 and the hyperboloid s_2, and this tangency must occur in the x-y plane.

To complete the canonical transformation generated by W, the P_i will be identified with h, α and β as follows:

$$
\begin{aligned}
P_1 &= P_h = h \\
P_2 &= P_\beta = \beta \\
P_3 &= P_\alpha = \alpha
\end{aligned}
\tag{55}
$$

The conjugate coordinates Q_i then become

$$
\begin{aligned}
Q_1 &= Q_h = \frac{\partial W}{\partial h} = \frac{c}{\sqrt{2}} \int_{r_1}^{q_1} \frac{q_1^2\, dq_1}{R} - \frac{c}{\sqrt{2}} \int_{s_2}^{q_2} \frac{q_2^2\, dq_2}{S} \\
Q_2 &= Q_\beta = \frac{\partial W}{\partial \beta} = -\frac{c}{\sqrt{2}} \int_{r_1}^{q_1} \frac{dq_1}{R} + \frac{c}{\sqrt{2}} \int_{s_2}^{q_2} \frac{dq_2}{S} \\
Q_3 &= Q_\alpha = \frac{\partial W}{\partial \alpha} = -\frac{\sqrt{2}\,\alpha}{c} \int_{r_1}^{q_1} \frac{dq_1}{(q_1^2 - 1) R} - \frac{\sqrt{2}\,\alpha}{c} \int_{s_2}^{q_2} \frac{dq_2}{(1 - q_2^2) S} + \varphi
\end{aligned}
\tag{56}
$$

In differentiating the integrals in W there are really three terms for each integral: one is the integral of the derivative of the integrand and the other two are obtained by evaluating the integrand at the limits and multiplying by the derivatives of the limits. The terms corresponding to the limits vanish, because the upper limits are not functions of h, α and β, the integrands for the q_1 and q_2 integrals vanish at the lower limits, and the lower limit of the φ integral is an absolute constant.

It will be noted that all the integrals occurring in Eq. (56) have forms identical with one or another of those occurring in Eqs. (44), (45) and (46) for the determination of q_1, q_2, t and φ as functions of u. The only difference is that in reference 6, where the integration of Eqs. (44), (45) and (46) is carried out in all detail, the lower limit on u was taken as zero. Here the lower limits are roots of R^2 and S^2.

Of the three Q_i, Q_β has a relatively simple interpretation if one replaces dq_1 and dq_2 by du in accordance with Eq. (44). Then Q_β becomes

$$Q_\beta = -\left[\frac{c}{\sqrt{2}} \int_{u(r_1)}^{u(q_1)} du \quad - \int_{u(s_2)}^{u(q_2)} du\right] \tag{57}$$

$$= \frac{c}{\sqrt{2}} \left(u\,(r_1) - u\,(s_2)\right)$$

since the upper limits correspond to a point on the orbit and therefore represent the same value of u. Thus Q_β appears proportional to the variation in u associated with a transit from tangency with a hyperboloid to tangency with an ellipsoid. Since the orbit is not, in general, periodic this statement does not yet uniquely define Q_β. To arrive at such a definition, it may be noted that in terms of the canonical variables P_i and Q_i the Hamiltonian becomes

$$H = h = P_1 \quad . \tag{58}$$

so that the Hamilton equations in these variables are:

$$\dot{P}_1 = \dot{P}_2 = \dot{P}_3 = \dot{h} = \dot{\alpha} = \dot{\beta} = 0 \tag{59}$$

and

$$\dot{Q}_\alpha = \dot{Q}_\beta = 0, \quad \dot{Q}_h = 1 \tag{60}$$

therefore

Q_α and Q_β are constants and

$$Q_h = t + \text{const} = t + C \tag{61}$$

The values of h, α and β may be obtained from a set of initial conditions. The values of Q_α, Q_β and C may be obtained from the initial conditions also, provided it is agreed that the $q_1 = r_1$ and $q_2 = s_2$ are to be associated, say, with the tangencies to the ellipsoid r_1 and the hyperboloid s_2 closest to the initial point. Other identifications of $q_1 = r_1$ and $q_2 = s_2$ will lead to Q's differing from those just defined by multiples of the periods K_1 and K_2.

If one applies the same analysis to Q_h and Q_α as used for Q_β' (replacing dq_1 and dq_2 by u), the following expressions are obtained:

$$Q_h = t + \frac{c}{\sqrt{2}} \left[\int_{u(r_1)}^{0} q_1^2 \, du - \int_{u(s_2)}^{0} q_2^2 \, du \right] \tag{62}$$

or

$$C = \frac{c}{\sqrt{2}} \left[\int_{u(r_1)}^{0} q_1^2 \, du - \int_{u(s_2)}^{0} q_2^2 \, du \right] \tag{63}$$

and

$$Q_\alpha = - \frac{\sqrt{2}\,\alpha}{c} \left[\int_{u(r_1)}^{0} \frac{du}{q_1^2 - 1} - \int_{u(s_2)}^{0} \frac{du}{1 - q_2^2} \right] \tag{64}$$

SECTION V - ACTION AND ANGLE VARIABLES

The action and angle variables are conventionally defined only for conditionally periodic systems, which means that for the two fixed center problem the development can be made only for $h < 0$. The action variables are defined in terms of the generating function W, as follows:

$$J_1 = \oint \frac{\partial W}{\partial q_1}\, d\,q_1 = \sqrt{2}\, c \oint \frac{R\, d\, q_1}{q_1^2 - 1}$$

$$J_2 = \oint \frac{\partial W}{\partial q_2}\, d\,q_2 = \sqrt{2}\, c \oint \frac{S\, d\, q_2}{1 - q_2^2} \qquad (65)$$

$$J_3 = \int_0^{2\pi} \frac{\partial W}{\partial q}\, d\varphi = 2\,\pi\,\alpha$$

where the integral for J_1 is taken over a complete cycle of variation of q_1 - i.e. from r_1 to r_2 and back to r_1, while that for J_2 is over a complete cycle of J_2 from s_1 to s_2 and back to s_1. These integrals can, for the most part, be reduced to the forms already encountered as follows:

$$J_1 = \sqrt{2}\, c \oint \frac{R\, d\, q_1}{q_1^2 - 1} = \sqrt{2}\, c \oint \frac{R^2}{q_1^2 - 1} \; \frac{d\, q_1}{R} \, .$$

$$= \sqrt{2}\, c \int_0^{4K_1} \left[h\, q^2 + \frac{\mu + \mu'}{c}\, q_1 - \beta - \frac{\alpha^2}{2c^2\,(q_1^2 - 1)} \right] du \qquad (66)$$

and its inverse is

$$J\begin{pmatrix} h & \beta & \alpha \\ J_1 & J_2 & J_3 \end{pmatrix} = \begin{bmatrix} \dfrac{1}{4K_1(n_1 - n_2)} & \dfrac{1}{4K_2(n_1 - n_2)} & \dfrac{m_1 + m_2}{2\pi\,(n_1 - n_2)} \\[2ex] \dfrac{\sqrt{2}\, n_2}{4cK_1(n_1 - n_2)} & \dfrac{\sqrt{2}\, n_1}{4cK_2(n_1 - n_2)} & \dfrac{\sqrt{2}\,(n_1 m_2 - m_1 n_2)}{2\pi\, c(n_1 - n_2)} \\[2ex] 0 & 0 & \dfrac{1}{2\pi} \end{bmatrix} \tag{74}$$

so that, finally

$$\begin{aligned} w_1 &= \frac{t + C}{4K_1\,(n_1 - n_2)} + \frac{\sqrt{2}\; n_2\, Q_\beta}{4cK_1\,(n_1 - n_2)} \\ w_2 &= \frac{t + C}{4K_2\,(n_1 - n_2)} + \frac{\sqrt{2}\; n_1\, Q_\beta}{4cK_2\,(n_1 - n_2)} \\ w_3 &= \frac{(t + C)\,(m_1 + m_2)}{2\pi\,(n_1 - n_2)} + \frac{\sqrt{2}\; Q_\beta\,(n_1 m_2 + m_1 n_2)}{2\pi\, c\,(n_1 - n_2)} + \frac{Q_\alpha}{2\pi} \end{aligned} \tag{75}$$

are the angle variables.

A complete cycle of variation q_1 corresponds to a variation in u of $4K_1$. Now the first term in this integral has the form of the dependence of the time on q_1, and, referring to Eq. (50) it is seen that the periodic part F_1 will vanish and hence the contribution of the first term to the integral is $8hn_1K_1$. Similarly the last term has the form of the q_1 part of the φ integral, Eq. (51), and will contribute $-\alpha \cdot 4m_1 K_1$. The β term contributes just $-\sqrt{2}\, c\, \beta \cdot 4K_1$. The only new integral to evaluate is

$$\int_0^{4K_1} q_1 \, du \tag{67}$$

This integral, too, turns out to be expressible as a linear term in u plus a periodic one, so that for the limits given, it contributes a term $\sqrt{2}\,(\mu + \mu')\,\ell_1 \cdot 4K_1$ where ℓ_1 is the coefficient of the linear term. Thus, finally,

$$J_1 = 8hn_1K_1 + 4\sqrt{2}\,(\mu + \mu')\,\ell_1 K_1 - 4\sqrt{2}\, c\, \beta K_1 - 4\alpha m_1 K_1 \tag{68}$$

In an exactly similar fashion

$$J_2 = -8hn_2K_2 + 4\sqrt{2}\,(\mu - \mu')\,\ell_2 K_2 + 4\sqrt{2}\, c\, \beta K_2 - 4\alpha m_2 K_2 \tag{69}$$

To obtain the angle variables conjugate to the action variables, it is necessary to recall that the original condition imposed on the P_i was only that they be constants. Identification of the P_i with h, α, and β is only one possibility; any three independent functions of h, α, and β would serve as well and, in particular, it is now desirable to identify P_i with J_i. Now the generating function W is given in Eq. (54) in terms of q_1, q_2, φ, h, α, β, and r_1 and s_2. The roots r_1 and s_2 are, however, functions of h, α and β. Now if Eqs. (68) and (69) together with the third of Eqs. (65) be inverted to express h, α, and β in terms of J_1, J_2, and J_3, it will be possible to substitute for h, α, and β in W to obtain W as a function of q_1, q_2, φ, J_1, J_2, and J_3. It should be remarked that the inversion to obtain h, α, and β

in terms of J_1, J_2 and J_3 is not an easy task since the coefficients n_1, n_2, ℓ_1, ℓ_2, m_1, m_2 are very complicated functions of h, α and β. Nevertheless the procedure is possible in principle and the angle variables w_i conjugate to the J's are given by the partial derivatives of the generating function W with respect to the J's:

$$w_i = \frac{\partial W}{\partial J_i} \tag{70}$$

One may obtain expressions for the w_i without actually performing the inversion, by writing the derivatives of W with respect to J_i in terms of its derivatives with respect to h, α, and β:

$$\begin{aligned} w_i &= \frac{\partial W}{\partial J_i} = \frac{\partial W}{\partial h}\frac{\partial h}{\partial J_i} + \frac{\partial W}{\partial \alpha}\frac{\partial \alpha}{\partial J_i} + \frac{\partial W}{\partial \beta}\frac{\partial \beta}{\partial J_i} \\ &= Q_h \frac{\partial h}{\partial J_i} + Q_\alpha \frac{\partial \alpha}{\partial J_i} + Q_\beta \frac{\partial \beta}{\partial J_i} \end{aligned} \tag{71}$$

from Eqs. (56) defining the variables conjugate to h, α and β. Or, recalling Eq. (62) for Q_h,

$$w_i = (t + C)\frac{\partial h}{\partial J_i} + Q_\beta \frac{\partial \beta}{\partial J_i} + Q_\alpha \frac{\partial \alpha}{\partial J_i} \tag{72}$$

where C, Q_α and Q_β are constants.

The derivatives of h, α and β may be expressed in terms of the n's, m's, ℓ's and K's occurring in Eqs. (68) and (69) by first obtaining the partials of the J's with respect to h, α and β from Eqs. (65), and then inverting their Jacobian matrix. The results of this calculation for the Jacobian are

$$J\begin{pmatrix} J_1 J_2 J_3 \\ h\ \beta\ \alpha \end{pmatrix} = \begin{bmatrix} 4\,n_1 K_1 & -2\sqrt{2}\,c K_1 & -4\,m_1 K_1 \\ -4\,n_2 K_2 & 2\sqrt{2}\,c K_2 & -4\,m_2 K_2 \\ 0 & 0 & 2\pi \end{bmatrix} \tag{73}$$

SECTION VI - CONCLUSION

To complete the solution of the restricted problem, it is now necessary to express the disturbing function H_2 in terms of the action and angle variables. This is a formidable problem. The disturbing function is given in terms of q_1, q_2, φ and their conjugate momenta in Eq. (34). The momenta are given in terms of q_1, q_2, φ, h, α and β by Eqs. (39) so that H_2 may readily be written in terms of these variables. Starting from the other end, the action variables J_1 and J_2 are given in terms of complicated functions of h, α, and β [Eqs. (68) and (69)] while J_3 is just $2\pi\alpha$ [Eqs. (65)]. The angle variables w_i are given by Eq. (75) as linear functions of Q_h, Q_α, and Q_β with coefficients which are functions of h, α, and β similar to those occurring for J_i. And Q_h, Q_α, and Q_β are related to q_1, q_2, φ, and h, α, and β by Eqs. (56). Thus, the following procedure would yield the information necessary to write H_2 (w_i, J_i):

1. Express K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, m_2 as functions of h, α, and β.

2. $$\alpha = \frac{J_3}{2\pi}$$

   Invert Eqs. (68) and (69) using the results of step 1 to obtain $h(J_i)$ and $\beta(J_i)$.

3. Express K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, m_2 which are functions of h, α and β in terms of J_i.

4. Invert Eqs. (75) to obtain $Q_h = t + c$, Q_α and Q_β as functions of the angle variables w_i and K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, and m_2.

5. Use step 1 to obtain Q_h, Q_α, and Q_β as functions of w_i and J_i.

6. Invert Eqs. (56) to obtain q_1, q_2, and φ as functions of Q_h, Q_α, Q_β, h, α, and β.

7. In the expressions for q_1, q_2, and φ obtained in step 6 replace Q_h, Q_α, and Q_β using step 5 and h, α, and β using step 2 to obtain q_1, q_2, and φ in terms of w_i and J_i.

8. In the disturbing function H_2 (q_1, q_2, φ, h, α, β), replace q_1, q_2, and φ from step 7 and h, α, and β from step 2 to obtain, finally, H_2 (w_i, J_i).

Steps 1, 2, and 6 are the difficult ones in this procedure. It is relatively easy to write K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, and m_2 as functions of the roots of the quartics and two intermediate parameters which are related to the roots of the quartics by transcendental equations. The roots of the quartics are, of course, functions of h, α, and β, but it is not easy to write out these functions explicitly. Thus, even step 1 is quite difficult, and to perform the inversion required in step 2 in closed form appears nearly impossible.

It should be remarked, however, that, at least for certain types of orbits, it should be possible to get fairly good approximations of these steps. For a lunar orbit which starts from the earth, closely circles the moon and returns to the earth, it may be shown that $\alpha^2/2c^2$ is very small. This is so because such an orbit has very close approaches to the line of centers, and recalling that α is the angular momentum about the line of centers, it follows that α must be small. If α were zero, two of the roots of the quartics would be ±1 and the other two are obtained in terms of h and β by solving quadratics [see Eqs. (40) and (41)]. Now it is possible to obtain the roots of the quartics for small α in terms of those for zero α in a series of powers of α. Thus for small α, it is easy to obtain fairly simple approximate expressions for the roots in terms of h, α, and β. Further, it turns out that the transcendental equations to be inverted for the intermediate parameters are very well approximated by just two terms of an expansion. Thus, it is feasible, for lunar orbits, to obtain a good approximation to steps 1 and 2.

The complete elliptic integrals

$$\oint q_1{}^2\, du, \qquad \oint \frac{d\,q_1}{q_1{}^2 - 1}, \qquad \oint d q_1$$

and similar ones for q_2, have forms very similar to those obtained by Vinti[8] in his model for the oblate earth. Vinti used oblate spheroidal coordinates for his model and the close connection between his development and that given in this report for the two fixed center problem was first pointed out by Pines[9]. The Vinti integrals have recently been evaluated approximately by Izsak[10] using a technique developed by Sommerfeld[11, 12] for evaluating certain contour integrals of functions with branch points. The method is to expand the integrals in terms of a quadratic function and evaluate the series of resulting integrals about contours enclosing the roots of the quadratic. The values of the integrals so obtained are explicitly in terms of the coefficients of the quartics. For the method to be valid, the expansion must converge over both the original and the final contours. This condition is satisfied for Izsak's expansion of the Vinti integrals. However, none of the obvious expansions for the two fixed center integrals converge over the final contour.

The greatest difficulty in following the procedure for obtaining H_2 is in step 6. Eqs. (56) relating Q_h, Q_α, and Q_β with q_1, q_2, and φ are transcendental equations and it is hard to say how well their inversion could be approximated by some approximation procedure, such as the Lagrange inversion theorem.

It should be remarked that it would be possible to write H_2 in terms of Q_h, Q_α , Q_β, h, α, and β rather than in terms of w_i and J_i. This is not done in the Kepler problem because the relation between the original coordinates and time is best achieved by a Fourier expansion in the mean anomaly rather than in time. An expansion in time would involve far more complicated coefficients. Which set of variables will turn out to be better for the two fixed center problem is hard to predict at this stage.

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Part II of
Second Semiannual Report

TWO-POINT BOUNDARY-VALUE PROBLEM OF THE CALCULUS OF VARIATIONS FOR OPTIMUM ORBITS

RAC 720-4

Research Regarding
Guidance and Space Flight Theory
Relative to the Rendezvous Problem
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Farmingdale, L. I., N. Y.

FOREWORD

This document is Part II of the Second Semiannual Report prepared by Republic Aviation Corporation under NASA Contract No. NAS 8-2605. The report will appear in slightly different format in "Progress Report No. 3 On Studies In The Fields Of Space Flight And Guidance Theory," issued by the Aeroballistics Division of Marshall Space Flight Center.

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CONTENTS

DEFINITION OF SYMBOLS

$\underline{R}$	Vehicle position vector
r	Distance to vehicle
$\underline{V}$	Velocity vector of vehicle
v	Speed of vehicle
$\underline{\xi}$	Perturbation displacement vector
f, g, f, g	Coordinate functions
μ	Mass parameter
t	Time
t_F	Time at which the natural end point is reached
k	Magnitude of thrust
$\underline{T}$	Direction of thrust
m	Mass of vehicle
λ, γ, σ	Lagrange multipliers or adjoint variables
a	Semi major axis
n	Mean motion
d_i	$\underline{R}_i \cdot \underline{\dot{R}}_i$
θ	Incremental eccentric anomaly
f_1, f_2, f_3, f_4	Functions of θ defined by Eqs. (48)
$\{\lambda\}$	Adjoint variables defined by Eq. (18)
$\{r\}$	State variables defined by Eq. (18)
$\{\delta(t_F)\}$	Residual vector defined by Eq. (19)
$\{\alpha\}$	Variational parameters
$\{p\}, \{q\}$	Defined by Eqs. (21), (22), (23)
$[\Phi]$	Partial derivatives of state variables as defined by Eq. (25)

$[\Lambda]$	Partial derivatives of adjoint variables as defined by Eq. (26)
$[F]$, $[G]$, $[J]$	Defined by Eqs. (27), (28), (29)

Subscripts

u	Unperturbed solution
o	Value at the initial time t_o
E	Value at the natural end point
A, B	Values corresponding to variational parameter set A or B

Superscripts

k	Value at the kth iteration

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TWO-POINT BOUNDARY-VALUE PROBLEM OF THE CALCULUS OF VARIATION FOR OPTIMUM ORBITS

By Jack Richman

SUMMARY

Abstract

This report contains a description for the solution of the two-point boundary-value problem of the calculus of variations for optimum orbits.

The method employed uses Lagrange multipliers and Pontryagin's maximum principle to obtain the decision functions.

In addition, two differential correction schemes are described. The first scheme is a "method by forward integration," and the second is an alternate "method by backward integration" that attempts to reduce the difficulties that might be encountered in inverting a differential correction matrix.

The optimum orbit is determined by a perturbation method similar to that of Encke and accommodates hyperbolic as well as elliptic orbits. The equations necessary for the generation of a digital-computer program are derived.

Author

INTRODUCTION

The usual methods of solving the two-point boundary-value problem of the calculus of variations involve the use of iterative gradient techniques. With these methods, the desired solution is reached only after making a great number of incremental variations and examining the changes that these variations cause. As one might expect, the rate of convergence for this method is very slow.

Another method of solving the two point boundary value problem of the calculus of variations, which will be described in this report, is one where all the decision functions and trajectories that are being used are extremals. This method uses, in addition to the state variables Lagrange multipliers or adjoint variables that play the key role in deciding the optimal direction of thrust, time of thrust duration, etc. The adjoint variables also define the natural end-point conditions by which the two-point boundary-value problem can be terminated. This natural end point, in general, will not be the desired end point. A differential correction scheme provides the means of obtaining another optimum trajectory the natural end point of which will be closer to the desired end point.

EQUATIONS OF MOTION

In a Newtonian system, the equations of motion of a particle that is in the gravitational field of N attracting bodies and is subject to other accelerations, such as thrust, drag, oblateness, radiation pressure, etc., are given by

$$\ddot{\underline{R}}_V = -\sum_{K=1}^{N} \mu_{B_K} \frac{\underline{R}_{VB_K}}{r^3_{VB_K}} + \sum_j \underline{F}_j \tag{1}$$

The problem that will be considered here is one in which the vehicle is in the gravitational field of only one body and is subjected to a variable thrust $\underline{k}$. In this case, Eq. (1) is reduced to

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}}{r^3} + \frac{k}{m} \underline{T} \tag{2}$$

where $\underline{T}$ is a unit vector in the direction of thrust. The magnitude of the thrust is taken to be proportional to the mass flow and is given by

$$k = -c\dot{m} \tag{3}$$

The constant of proportionality c is related to the more commonly used constant specific impulse I_{sp} by

$$c = I_{sp} g \tag{4}$$

DERIVATION OF OPTIMIZATION EQUATIONS

In the derivation of the optimization equations, it will be assumed that the vehicle can have two possible values of thrust, either $k = k_{max}$ or $k = k_{min}$. The magnitudes of these two thrust values may differ with each stage.

Minimum-Fuel Condition

The value of the integral to be minimized is given by

$$I = \int_{t_o}^{t_F} -dm = \int_{t_o}^{t_F} -\dot{m}\, dt \tag{5}$$

and the conditions of constraint are given by

$$\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} - \frac{k}{m}\underline{T} = 0$$

$$\dot{\underline{R}} - \underline{V} = 0 \tag{6}$$

$$\dot{m} + \frac{k}{c} = 0$$

Because these conditions of constraint are satisfied at every point on the trajectory, we may rewrite Eq. (5), without changing its value, as

$$I = \int_{t_o}^{t_F} \left[-\dot{m} + \underline{\lambda} \cdot (\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} - \frac{k}{m}\underline{T}) + \underline{\gamma} \cdot (\dot{\underline{R}} - \underline{V}) + \sigma (\dot{m} + \frac{k}{c}) \right] dt$$

$$= \int_{t_o}^{t_F} L\,(\dot{\underline{R}}, \underline{R}, \dot{\underline{V}}, \underline{V}, \dot{m}, m, \underline{\lambda}, \underline{\gamma}, \sigma)\, dt \tag{7}$$

where $\underline{\lambda}(t)$, $\underline{\gamma}(t)$, and $\sigma(t)$ are undetermined Lagrange multipliers that are chosen so as to determine the optimum decision functions required to solve the problem.

Applying the Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{8}$$

to the state variables, results in the following set of equations:

$$\dot{\underline{\lambda}} + \underline{\gamma} = 0$$

$$\dot{\underline{\gamma}} - \frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} \underline{R} = 0 \tag{9}$$

$$\dot{\sigma} - \frac{k}{m^2} \underline{\lambda} \cdot \underline{T} = 0$$

Equations (6) and (9) can be combined to form

$$
\begin{aligned}
\ddot{\underline{R}} &= -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\underline{T} \\
\ddot{\underline{\lambda}} &= -\frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^5}\underline{R} \\
k &= -c\dot{m} \\
\dot{\sigma} &= \frac{k}{m^2}\underline{\lambda}\cdot\underline{T}
\end{aligned}
\tag{10}
$$

In addition, the natural boundary conditions are

$$
\begin{aligned}
\underline{\lambda}\cdot\delta\underline{V}\Big|_{t_o}^{t_F} &= 0 \\
\underline{\gamma}\cdot\delta\underline{R}\Big|_{t_o}^{t_F} &= 0 \\
(\sigma-1)\delta m\Big|_{t_o}^{t_F} &= 0
\end{aligned}
\tag{11}
$$

Because variations in the position and velocity at the end points are zero, the first two expressions of Eq. (11) yield no additional information about the values of $\underline{\lambda}$ and $\underline{\gamma}$ at the end points. The variation of mass at the final end point, however, is not zero, i.e., $\delta m(t_F) \neq 0$. Hence, the only way to satisfy the third expression of Eq. (11) is to demand that

$$
\sigma(t_F) - 1 = 0 \tag{12}
$$

The only additional information that is necessary to completely define the extremal is the determination of the optimum thrust vector and the duration of this thrust.

For the determination of this decision function, we make use of Pontryagin's "Maximum Principle," [1,2] which states that a necessary condition for an integral of the form of Eq. (7) to be minimized is that the Hamiltonian be a maximum. The Hamiltonian for this problem is given by

$$H = \underline{\lambda} \cdot \dot{\underline{V}} + \underline{\gamma} \cdot \underline{V} + \sigma \dot{m}$$

$$= \underline{\lambda} \cdot \left[-\frac{\mu}{r^3} \underline{R} + \frac{k}{m} \underline{T} \right] - \dot{\underline{\lambda}} \cdot \dot{\underline{R}} - \frac{\sigma k}{c} \tag{13}$$

$$= \left[-\frac{\mu \underline{R} \cdot \underline{\lambda}}{r^3} - \dot{\underline{\lambda}} \cdot \dot{\underline{R}} \right] + k \left[\frac{\underline{\lambda} \cdot \underline{T}}{m} - \frac{\sigma}{c} \right]$$

For H to be a maximum, the unit thrust vector $\underline{T}$ must be in the direction of $\underline{\lambda}$, or

$$\underline{T} = \frac{\underline{\lambda}}{|\underline{\lambda}|} \tag{14}$$

Therefore, the coefficient of k in Eq. (13), which is defined as the switch function, becomes

$$S = \frac{|\underline{\lambda}|}{m} - \frac{\sigma}{c} \tag{15}$$

The necessary conditions that must be placed on the magnitude of the thrust for H to be a maximum are the following:

$$\begin{aligned} &\text{if } S > 0 \quad \text{then} \quad k = k_{max} \\ &\text{if } S < 0 \quad \text{then} \quad k = k_{min} \end{aligned} \tag{15a}$$

Furthermore, when thrust is applied, it is desirable to make the switch function as large as possible. This can be accomplished by allowing the mass to be as small as permissible, which implies the obvious condition that any empty tanks or other unnecessary weight be dropped as soon as possible.

<u>Minimum-Time Condition</u>

In this case, the value of the integral to be minimized is given by

$$I = \int_{t_o}^{t_F} dt = \int_{t_o}^{t_F} \left[1 + \underline{\lambda} \cdot (\dot{\underline{V}} + \frac{\mu \underline{R}}{r^3} - \frac{k}{m} \underline{T}) + \underline{\gamma} \cdot (\dot{\underline{R}} - \underline{V}) + \sigma (\dot{m} + \frac{k}{c}) \right] dt \tag{16}$$

Application of the Euler Lagrange equations and Pontryagin's Principle lead to the exact same results as the minimum-fuel condition, with the exception of one of the natural-boundary conditions. In place of the third expression in Eq. (11), we now have

$$\sigma\,\delta m \Big|_{t_o}^{t_F} = 0$$

or

$$\sigma\,(t_F) = 0 \tag{17}$$

Therefore, for the "minimum-time" condition the natural end point occurs when $\sigma = 0$.

ITERATION SCHEME

General Procedure

The problem is to generate a set of initial adjoint variables such that an optimum orbit can be computed where the natural end point matches the desired end point. (The end points are, of course, given by terminal values of the state variables.) With initial values of the state variables specified and an estimate for the initial values of the adjoint variables, an iterative method can be used to solve this problem. Improved estimates for the initial values of the adjoint variables can be obtained by computing the residuals or differences between the values of the state variables at the desired end point and the natural end point and then applying a differential correction matrix to these residuals. We define the $\{r\}$, $\{\lambda\}$, and $\{\delta\,(t_F)\}$ vectors as

$$\{r\} = \begin{Bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ m \end{Bmatrix} , \qquad \{\lambda\} = \begin{Bmatrix} -\dot{\lambda}_x \\ -\dot{\lambda}_y \\ -\dot{\lambda}_z \\ \lambda_x \\ \lambda_y \\ \lambda_z \\ \sigma \end{Bmatrix} , \tag{18}$$

and

$$\{\delta(t_F)\} = \begin{Bmatrix} x\,(t_F) - x_E \\ y\,(t_F) - y_E \\ z\,(t_F) - z_E \\ \dot{x}\,(t_F) - \dot{x}_E \\ \dot{y}\,(t_F) - \dot{y}_E \\ \dot{z}\,(t_F) - \dot{z}_E \\ m(t_F) - m_E \end{Bmatrix} \tag{19}$$

where the subscript E denotes the values of the state variables at the desired end point.

The Kth approximation to $\{\lambda(t_o)\}$ is designated by $\{\lambda^{(K)}(t_o)\}$, and it is desired to obtain an improved value of $\{\lambda^{(K+1)}(t_F)\}$. The procedure is as follows: using $\{\lambda^{(K)}(t_o)\}$ in the integration scheme, the position, velocity and mass at time t_F, as well as the residuals $\{\delta^{(K)}(t_F)\}$, are computed; and the initial values of the adjoint variables are then changed so as to reduce the residuals,

$$\left\{\lambda^{(K+1)}(t_o)\right\} = \left\{\lambda^{(K)}(t_o)\right\} + \left\{\Delta\lambda^{(K)}(t_o)\right\} \tag{20}$$

where $\{\Delta\lambda^{(K)}(t_o)\}$ is to be found by using a differential correction matrix.

Methods for Obtaining the Differential Correction Matrix

Making use of Eqs. (14) and (18), the first two expressions of Eq. (10) can be written as follows:

$$\begin{aligned} \{\dot{r}\} &= \left\{q(\{r\},\{\lambda\})\right\} \quad \text{or} \quad \dot{r}_i = q_i(\{r\},\{\lambda\}) \\ \{\dot{\lambda}\} &= \left\{p(\{r\},\{\lambda\})\right\} \quad \text{or} \quad \dot{\lambda}_i = p_i(\{r\},\{\lambda\}) \end{aligned} \tag{21}$$

where

$$\begin{aligned} q_1 &= \dot{x} \\ q_2 &= \dot{y} \\ q_3 &= \dot{z} \\ q_4 &= -\frac{\mu x}{r^3} + \frac{k}{m}\frac{\lambda_x}{|\lambda|} \\ q_5 &= -\frac{\mu y}{r^3} + \frac{k}{m}\frac{\lambda_y}{|\lambda|} \\ q_6 &= -\frac{\mu z}{r^3} + \frac{k}{m}\frac{\lambda_z}{|\lambda|} \\ q_7 &= -\frac{k}{c} \end{aligned} \tag{22}$$

and

$$p_1 = \frac{\mu \lambda_x}{r^3} - \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} x$$

$$p_2 = \frac{\mu \lambda_y}{r^3} - \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} y$$

$$p_3 = \frac{\mu \lambda_z}{r^3} - \frac{3\mu (\underline{R} \cdot \underline{\lambda})}{r^5} z$$

$$p_4 = \dot{\lambda}_x \tag{23}$$

$$p_5 = \dot{\lambda}_x$$

$$p_6 = \dot{\lambda}_z$$

$$p_7 = \frac{k}{m^2} \; |\lambda|$$

Taking the variations of Eq. (21) with respect to a set of parameters $\{\alpha\} = (\alpha_1, \alpha_2, \ldots\ldots \alpha_7)$, we find that

$$\frac{d}{dt} [\Phi] = [F] [\Phi] + [G] [\Lambda] \tag{24}$$

$$\frac{d}{dt} [\Lambda] = -[F]^* [\Lambda] + [J] [\Phi]$$

where

$$
[\Phi] = \frac{\partial\{r\}}{\partial\{\alpha\}} \equiv \begin{bmatrix}
\frac{\partial x(t)}{\partial \alpha_1} & \frac{\partial x(t)}{\partial \alpha_2} & \frac{\partial x(t)}{\partial \alpha_3} & \frac{\partial x(t)}{\partial \alpha_4} & \frac{\partial x(t)}{\partial \alpha_5} & \frac{\partial x(t)}{\partial \alpha_6} & \frac{\partial x(t)}{\partial \alpha_7} \\
\frac{\partial y(t)}{\partial \alpha_1} & & & & & & \\
\frac{\partial z(t)}{\partial \alpha_1} & & & & & & \\
\frac{\partial \dot{x}(t)}{\partial \alpha_1} & & & & & & \\
\frac{\partial \dot{y}(t)}{\partial \alpha_1} & & & & & & \\
\frac{\partial \dot{z}(t)}{\partial \alpha_1} & & & & & & \\
\frac{\partial m(t)}{\partial \alpha_1} & & & & & & \frac{\partial m(t)}{\partial \alpha_7}
\end{bmatrix} \tag{25}
$$

$$
[\Lambda] = \frac{\partial\{\lambda\}}{\partial\{\alpha\}} \equiv \begin{bmatrix}
\frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_1} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_2} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_3} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_4} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_5} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_6} & \frac{-\partial\dot{\lambda}_x(t)}{\partial\alpha_7} \\
\frac{-\partial\dot{\lambda}_y(t)}{\partial\alpha_1} & & & & & & \\
\frac{-\partial\dot{\lambda}_y(t)}{\partial\alpha_1} & & & & & & \\
\frac{\partial\lambda_x(t)}{\partial\alpha_1} & & & & & & \\
\frac{\partial\lambda_y(t)}{\partial\alpha_1} & & & & & & \\
\frac{\partial\lambda_z(t)}{\partial\alpha_1} & & & & & & \\
\frac{\partial\sigma(t)}{\partial\alpha_1} & & & & & & \frac{\partial\sigma(t)}{\partial\alpha_7}
\end{bmatrix} \tag{26}
$$

$$[F] = \frac{\partial\{q\}}{\partial\{r\}} \equiv \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-\mu}{r^3} + \frac{3\mu x^2}{r^5} & \frac{3\mu xy}{r^5} & \frac{3\mu xz}{r^5} & 0 & 0 & 0 & \frac{-k\lambda_x}{m^2\lambda} \\ \frac{3\mu xy}{r^5} & \frac{-\mu}{r^3} + \frac{3\mu y^2}{r^5} & \frac{3\mu yz}{r^5} & 0 & 0 & 0 & \frac{-k\lambda_y}{m^2\lambda} \\ \frac{3\mu xz}{r^5} & \frac{3\mu yz}{r^5} & \frac{-\mu}{r^3} + \frac{3\mu z^2}{r^5} & 0 & 0 & 0 & \frac{-k\lambda_z}{m^2\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (27)$$

$$[G] = \frac{\partial\{q\}}{\partial\{\lambda\}} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{m\lambda} - \frac{k\lambda_x^2}{m\lambda^3} & \frac{-k\lambda_x\lambda_y}{m\lambda^3} & \frac{-k\lambda_x\lambda_z}{m\lambda^3} & 0 \\ 0 & 0 & 0 & \frac{-k\lambda_x\lambda_y}{m\lambda^3} & \frac{k}{m\lambda} - \frac{k\lambda_y^2}{m\lambda^3} & \frac{-k\lambda_y\lambda_z}{m\lambda^3} & 0 \\ 0 & 0 & 0 & \frac{-k\lambda_x\lambda_z}{m\lambda^3} & \frac{-k\lambda_y\lambda_z}{m\lambda^3} & \frac{k}{m\lambda} - \frac{k\lambda_z^2}{m\lambda^3} & 0 \end{bmatrix} \quad (28)$$

$$
[J] = \frac{\partial\{p\}}{\partial\{r\}} \equiv - \begin{bmatrix}
6\mu\lambda_x x + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^5}(1 - \frac{5x^2}{r^2}) & \frac{3\mu}{r^5}(\lambda_x y + x\lambda_y) - \frac{15\mu}{r^7}xy(\underline{R}\cdot\underline{\lambda}) & \frac{3\mu}{r^5}(\lambda_x z + x\lambda_z) - \frac{15\mu}{r^7}xz(\underline{R}\cdot\underline{\lambda}) & 0 & 0 & 0 \\
\frac{3\mu}{r^5}(\lambda_y x + y\lambda_x) - \frac{15\mu}{r^7}xy(\underline{R}\cdot\underline{\lambda}) & \frac{6\mu\lambda_y y}{r^5} + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^5}(1 - \frac{5y^2}{r^2}) & \frac{3\mu}{r^5}(\lambda_y z + y\lambda_z) - \frac{15\mu}{r^7}yz(\underline{R}\cdot\underline{\lambda}) & 0 & 0 & 0 \\
\frac{3\mu}{r^5}(\lambda_z x + z\lambda_x) - \frac{15\mu}{r^7}xz(\underline{R}\cdot\lambda) & \frac{3\mu}{r^5}(\lambda_z y + z\lambda_y) - \frac{15\mu}{r^7}yz(\underline{R}\cdot\underline{\lambda}) & \frac{6\mu\lambda_z z}{r^5} + \frac{3\mu(\underline{R}\cdot\underline{\lambda})}{r^3}(1 - \frac{5z^2}{r^5}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{29}
$$

Method of Forward Integration. Two convenient sets of parameters to work with are the sets that consist of the initial values of the state variables and adjoint variables, which are, respectively

$$\{\alpha\}_A = \{r(t_o)\}$$

and

$$\{\alpha\}_B = \{\lambda(t_o)\} \tag{29}$$

Using these sets of parameters, Eq. (24) can be integrated "forward" simultaneously with equations of motion, using the initial values of $[\Phi]$ and $[\Lambda]$ as given by

$$\left\{\begin{array}{l}\left[\Phi_A(t_o)\right] = I \\ \\ \left[\Lambda_A(t_o)\right] = 0\end{array}\right\} \text{ and } \left\{\begin{array}{l}\left[\Phi_B(t_o)\right] = 0 \\ \\ \left[\Lambda_B(t_o)\right] = I\end{array}\right\} \tag{30}$$

The differential corrections are obtained by solving the system of equations

$$\begin{aligned}\{\Delta r(t_F)\} &= \left[\Phi_A(t_F)\right]\{\Delta r(t_o)\} + \left[\Phi_B(t_F)\right]\{\Delta\lambda(t_o)\} \\ \{\Delta\lambda(t_F)\} &= \left[\Lambda_A(t_F)\right]\{\Delta r(t_o)\} + \left[\Lambda_B(t_F)\right]\{\Delta\lambda(t_o)\}\end{aligned} \tag{31}$$

and, because

$$\{\Delta r(t_o)\} = 0 \quad \text{and} \quad \{\Delta r(t_F)\} = \{\delta(t_F)\}$$

we find

$$\{\Delta\lambda(t_o)\} = \left[\Phi_B(t_F)\right]^{-1}\{\delta(t_F)\} \tag{32}$$

An interesting feature of this differential correction scheme is a tendency for the inverse of the differential correction matrix $[\Phi_B(t_F)]$ to become more and more singular as the time arc increases. This tendency toward singularity is a problem of utmost interest.

Method of Backward Integration. If the use of double-precision techniques fails to provide the required numerical accuracy for the inverse of the matrix, an alternate method of generating the differential correction matrix can be used. This alternate scheme employs a method of "backward" integration to provide a differential correction matrix consisting of the sum of two matrices, only one of which requires inversion to produce the differential corrections. In this case, the two sets of parameters consist of the final values of the state variables and adjoint variables, which are, respectively

$$\{\alpha\}_A = \{r(t_F)\}$$

and (33)

$$\{\alpha\}_B = \{\lambda(t_F)\}$$

Using these sets of parameters, the variational Eq. (24) can be integrated "backward." The procedure is as follows: the equations of motion are integrated "forward" until the natural end point is reached; the residuals are computed; and, then, Eq. (24), together with the equations of motion, are simultaneously integrated "backward" starting at time t_F and ending at time t_o, using for initial values of $[\Phi]$ and $[\Lambda]$:

$$\left\{\begin{matrix} [\Phi_A(t_F)] = I \\ [\Lambda_A(t_F)] = 0 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} [\Phi_B(t_F)] = 0 \\ [\Lambda_B(t_F)] = I \end{matrix}\right\} \tag{34}$$

The differential corrections are obtained by solving the equations

$$\begin{aligned} \{\Delta\lambda(t_o)\} &= [\Lambda_A(t_o)]\{\Delta r(t_F)\} + [\Lambda_B(t_o)]\{\Delta\lambda(t_F)\} \\ \{\Delta r(t_o)\} &= [\Phi_A(t_o)]\{\Delta r(t_F)\} + [\Phi_B(t_o)]\{\Delta\lambda(t_F)\} \end{aligned} \tag{35}$$

and, because, in this case,

$$\{\Delta r(t_o)\} = 0 \quad \text{and} \quad \{\Delta r(t_F)\} = \{\delta(t_F)\}$$

solving Eq. (35) for $\{\Delta\lambda(t_o)\}$, we find that

$$\{\Delta\lambda(t_o)\} = \left[[\Lambda_A(t_o)] - [\Lambda_B(t_o)][\Phi_B(t_o)]^{-1}[\Phi_A(t_o)]\right]\{\delta(t_F)\} \tag{36}$$

Convergence of Iteration

Several difficulties are connected with the above iteration scheme, and some of them might be crucial enough to cause divergence of the iteration. These difficulties might arise for the following reasons:

1. In the variational equations, the variation of burning time is not accounted for.

2. The inversion of a matrix is required in both methods to obtain the differential-correction matrix. Furthermore this inversion becomes more involved since the residual $m(t_F)-m_E$ of the vector $\{\delta(t_F)\}$ is unspecified and requires additional computation.

3. The change Δt_F in the final time has not been taken into account. However, this should be included by considering the additional transversality condition which results in $\{\lambda\} \cdot \{r\} = 0$.

DIGITAL PROGRAM

Trajectory Equations

The equations that completely define the trajectory have been described previously. The order in which these equations are programmed for the general case (with thrust) is as follows:

$$s = \left(\frac{|\lambda|}{m} - \frac{\sigma}{c}\right) \qquad \begin{array}{ll} >0 & k = k_{max} \\ <0 & k = k_{min} \end{array}$$

$$\dot{m} = -\frac{k}{c}$$

$$\dot{\sigma} = \frac{k|\lambda|}{m^2}$$

$$\frac{d}{dt}[\Phi] = [F][\Phi] + [G][\Lambda]$$

$$\frac{d}{dt}[\Lambda] = -[F]^*[\Lambda] + [J][\Phi] \tag{37}$$

$$\ddot{\underline{R}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\frac{\underline{\lambda}}{|\lambda|}$$

$$\ddot{\underline{\lambda}} = -\frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu(\underline{\lambda} \cdot \underline{R})}{r^5}\underline{R}$$

$$m = m(t) + \int_t^{t+\Delta t} \dot{m}\, dt$$

$$\sigma = \sigma(t) + \int_t^{t+\Delta t} \dot{\sigma}\, dt$$

These equations are integrated until the natural end point is reached. At that time, the residuals are computed and compared to a predetermined set of maximum permissible values $\{\epsilon\}$.

If $\delta_j(t_F) \leq \epsilon_j$ for all the residuals, then that trajectory is the solution to the two-point boundary-value problem. If $\delta_j(t_F) > \epsilon_j$ for any of the residuals, then a differential correction is applied to the initial values of the adjoint variables as described previously. If the alternate differential correction scheme is used, then a "backward" integration is necessary before any corrections can be applied.

Numerical Procedures

The differential equations of Eq. (37) can be integrated numerically with a Runge-Kutta fourth-order method. To reduce any accumulation of error that might result from a number of step-by-step integration, however, it is convenient to write the equation of motion for the high thrust case in the form

$$\ddot{\underline{R}} = \ddot{\underline{R}}_u + \ddot{\underline{\xi}} \tag{38a}$$

The velocity and position vectors can be written as

$$\begin{aligned} \dot{\underline{R}} &= \dot{\underline{R}}_u + \dot{\underline{\xi}} \\ \underline{R} &= \underline{R}_u + \underline{\xi} \end{aligned} \tag{38b}$$

where $\ddot{\underline{R}}_u$ is the unperturbed solution and $\underline{\xi}$ is the perturbation.

In this method, $\ddot{\underline{R}}_u$ is taken as

$$\underline{R}_u = \frac{k}{m}\,\underline{T}_i = -\frac{c\dot{m}}{m}\,\underline{T}_i \tag{39}$$

and

$$\ddot{\underline{\xi}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m}\left[\underline{T} - \underline{T}_i\right] \tag{40}$$

Eq. (40) is integrated numerically, and the solution to Eq. (39) is

$$\begin{aligned} \underline{R}_u &= f\,\underline{R}(t_i) + g\,\dot{\underline{R}}(t_i) + h\underline{T}(t_i) \\ \dot{\underline{R}}_u &= \dot{f}\,\underline{R}(t_i) + \dot{g}\,\dot{\underline{R}}(t_i) + \dot{h}\underline{T}(t_i) \end{aligned} \tag{41}$$

where

$$f = 1$$

$$g = t - t_i$$

$$h = -c\left\{\frac{1}{\dot{m}}\left[m \log m - m_i \log m_i - (m - m_i)\right] - (t - t_i) \log m_i\right\}$$

$$\dot{f} = 0$$

$$\dot{g} = 1$$

$$\dot{h} = - c(\log m - \log m_i)$$

$$m = m_i + (t - t_i)\, \dot{m}$$

(the subscript i refers to values at time t_i)

This perturbation method, or Encke scheme as it is commonly called, will reduce inaccuracies occurring in numerical integration, provided that the perturbation terms are small compared with the total solution. Whenever these perturbations become too large, a rectification takes place, i.e., an initialization occurs in which the values of the variable at time t now becomes the values of the variable at time t_i. A rectification takes place whenever any of the following conditions occur:

$$\left|\frac{\xi}{r}\right| > \epsilon_{pos} \qquad \text{(position rectification)}$$

$$\left|\frac{\dot{\xi}}{\dot{r}}\right| > \epsilon_{vel} \qquad \text{(velocity rectification)} \tag{42}$$

$$\sqrt{2\,\underline{T} \cdot \underline{T}_i} > \epsilon_{acc} \qquad \text{(acceleration rectification)}$$

SOLUTION OF EQUATIONS FOR THE COASTING STAGES

The solution of the equations of motion and the Euler Lagrange equations can be derived in closed form for the coasting period. In the no thrust region (k=0), the equation of motion reduces to

$$\ddot{\underline{R}} = -\frac{\mu \underline{R}}{r^3} \qquad \text{(Kepler problem)} \tag{43}$$

The two-body orbit that results from the solution of Eq. (43) with the initial conditions

$$\underline{R}(t_i) = \underline{R}_i$$
$$\dot{\underline{R}}(t_i) = \dot{\underline{R}}_i \tag{44}$$

can be written as a linear combination of $\underline{R}_i$ and $\dot{\underline{R}}_i$ as

$$\underline{R} = f\,\underline{R}_i + g\,\dot{\underline{R}}_i$$
$$\dot{\underline{R}} = \dot{f}\,\underline{R}_i + \dot{g}\,\dot{\underline{R}}_i \tag{45}$$

The coefficients f, g, $\dot{f}$, and $\dot{g}$ are obtained as follows: we represent the initial conditions by the set of elements

$$a = \left(\frac{2}{r_i} - \frac{v_i^2}{\mu}\right)^{-1}$$
$$d_i = \underline{R}_i \cdot \dot{\underline{R}}_i$$
$$n = \frac{\mu^{1/2}}{a^{3/2}} \qquad \text{(elliptic)} \tag{46}$$
$$n = \frac{\mu^{1/2}}{(-a)^{3/2}} \qquad \text{(hyperbolic)}$$

This results in the following Kepler's equation

$$n(t - t_i) = \theta - \sin\theta + \frac{r_i}{a}\sin\theta + \frac{d_i}{\sqrt{\mu a}}(1 - \cos\theta) \quad \text{(elliptic)}$$
$$n(t - t_i) = \sinh\theta - \theta - \frac{r_i}{a}\sinh\theta + \frac{d_i}{\sqrt{-\mu a}}(\cosh\theta - 1) \quad \text{(hyperbolic)} \tag{47}$$

where $\theta(t)$ is the incremental eccentric anomaly $E - E_i$; the functions f_1, f_2 f_3, f_4 are defined as

$$
\left.
\begin{array}{l}
\left.
\begin{array}{l}
f_1(\theta) = \theta - \sin\theta \\
f_2(\theta) = 1 - \cos\theta \\
f_3 = \sin\theta = \theta - f_1(\theta) \\
f_4 = \cos\theta = 1 - f_2(\theta)
\end{array}
\right\} \text{(elliptic)} \\
\\
\left.
\begin{array}{l}
f_1(\theta) = \sinh\theta - \theta \\
f_2(\theta) = \cosh\theta - 1 \\
f_3(\theta) = \sinh\theta = \theta + f_1(\theta) \\
f_4(\theta) = \cosh\theta = 1 + f_2(\theta)
\end{array}
\right\} \text{(hyperbolic)}
\end{array}
\right. \qquad (48)
$$

and the solution of the two-body problem for both elliptic and hyperbolic orbits is given by

$$
\begin{aligned}
f &= -\frac{|a|}{r_i} f_2 + 1 \\
g &= -\frac{1}{n} f_1 + (t - t_i) \\
\frac{r}{|a|} &= f_2 + \frac{r_i}{|a|} f_4 + \frac{d_i}{\sqrt{\mu |a|}} f_3 \\
\dot{f} &= -\sqrt{\frac{k}{|a|}} \frac{1}{r_i} \frac{|a|}{r} f_3 \\
\dot{g} &= -\frac{|a|}{r} f_2 + 1 \\
n(t - t_i) &= f_1 + \frac{r_i}{|a|} f_3 + \frac{d_i}{\sqrt{\mu |a|}} f_2
\end{aligned}
\qquad (49)
$$

For the non-thrust case, we also can solve for $\{\lambda\}$ in closed form. The following is a derivation leading to this closed-form solution: the differential equation for the adjoint variables are written as

$$
\frac{d}{dt} \{\lambda\} = -[F]^* \{\lambda\} \qquad (50)
$$

where $[F]$ is defined by Eq. (27); the variational equation for $[\Phi]$ reduces to

$$\frac{d}{dt}[\Phi] = [F][\Phi] \tag{51}$$

taking the transpose of Eq. (50) and postmultiplying by $[\Phi]$, yields

$$\frac{d}{dt}\{\lambda\}^*[\Phi] = -\{\lambda\}^*[F][\Phi] \tag{52}$$

premultiplying Eq. (51) by $\{\lambda\}^*$, yields

$$\{\lambda\}^*\frac{d}{dt}[\Phi] = \{\lambda\}^*[F][\Phi] \tag{53}$$

comparing Eqs. (52) and (53), we see that

$$\{\lambda\}^*\frac{d}{dt}[\Phi] + \frac{d}{dt}\{\lambda\}^*[\Phi] = 0$$

or

$$\frac{d}{dt}\left[\{\lambda\}^*[\Phi]\right] = 0 \tag{54}$$

Eq. (54) states that $\{\lambda\}^*[\Phi]$ is a constant and, therefore, can be written as

$$\{\lambda(t)\}^*[\Phi(t)] = \{\lambda(t_K)\}^*\left[\Phi(t_K)\right] \tag{55}$$

where t_K is any fixed time in the no-thrust interval; solving Eq. (55) for $\{\lambda(t)\}$, results in

$$\{\lambda(t)\} = \left[\Phi^*(t)\right]^{-1}\left[\Phi^*(t_K)\right]\left\{\lambda(t_K)\right\} \tag{56}$$

In the case where the set of parameters $\{\alpha\}$ corresponds to a set of the state variables $\{r\}$, the matrix $[\Phi]$ can be written as

$$\left[\Phi_A(t)\right] = \left[\Phi_A(t - t_K)\right]\left[\Phi_A(t_K)\right] \tag{57}$$

taking the transpose and then the inverse of Eq. (57), leads to

$$\left[\Phi^*(t)\right]^{-1} = \left[\Phi_A^*(t - t_K)\right]^{-1}\left[\Phi_A^*(t_K)\right]^{-1} \tag{58}$$

and combining Eqs. (56) and (58), results in

$$\{\lambda(t)\} = \left[\Phi_A^*(t - t_K)\right]^{-1} \{\lambda(t_K)\} \tag{59}$$

which is the closed form solution of $\{\lambda(t)\}$.

The elements of the $\left[\Phi_A(t - t_K)\right]$ matrix are obtained by differentiating the Kepler orbit elements with respect to $\underline{R}(t_K)$ and $\dot{\underline{R}}(t_K)$. The elements of $\left[\Phi_A(t - t_K)\right]_{pq} = \dfrac{\partial r_p(t)}{\partial r_q(t_K)}$, with p, q=1, ..., 7, are as follows:

$$\begin{aligned}
\frac{\partial x_i(t)}{\partial x_j(t_K)} \equiv \frac{\partial x_i}{\partial x_{oj}} &= f\,\delta_{ij} + \frac{3|a|}{r_o^3} x_{oj}\left[x_i - x_{oi} - (t - t_K)\dot{x}_{oi}\right] \\
&+ \frac{|a|x_{oj}}{r_o^3}(\dot{x}_i - \dot{x}_{oi})\left[-3(t - t_K) + g + \frac{r_o}{|a|n}\left(1 - \frac{r_o}{|a|}\right) f_3\right] \\
&- \frac{|a|}{\mu} f_2(\dot{x}_i - \dot{x}_{oi})\dot{x}_{oj} + f_2\left(\frac{1}{r_o} + \frac{1}{|a|}\right)\frac{a^2}{r_o^3} x_{oi}x_{oj}
\end{aligned} \tag{60}$$

$$\begin{aligned}
\frac{\partial x_i(t)}{\partial \dot{x}_j(t_K)} \equiv \frac{\partial x_i}{\partial \dot{x}_{oj}} &= g\,\delta_{ij} + \frac{3|a|}{\mu}\dot{x}_{oj}\left[x_i - x_{oi} - (t - t_K)\dot{x}_{oi}\right] \\
&+ \frac{|a|\dot{x}_{oj}}{\mu}(\dot{x}_i - \dot{x}_{oi})\left[-3(t - t_K) + g + \frac{r_o}{|a|n} f_3\right] \\
&- \frac{|a|}{\mu} x_{oj}(\dot{x}_i - \dot{x}_{oi})\, f_2 + \frac{a^2}{\mu r_o} f_2\, x_{oi}\dot{x}_{oj}
\end{aligned}$$

$$\frac{\partial \dot{x}_i(t)}{\partial x_j(t_K)} \equiv \frac{\partial \dot{x}_i}{\partial x_{oj}} = \dot{f}\,\delta_{ij} - \frac{\mu |a| x_{oj}}{r^3 r_o^3}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right]\left[-3(t-t_K) + g + \frac{r_o}{|a| n}\left(1 - \frac{r_o}{|a|}\right)f_3\right]$$

$$+ \frac{|a|\dot{x}_{oj}}{r^3}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right]f_2 + \frac{r_o}{\mu}(\dot{x}_i - \dot{x}_{oi})\,\dot{x}_{oj}\dot{f}$$

$$- \frac{|a|}{r_o^2}\dot{f}\left(\frac{1}{r_o} + \frac{1}{|a|}\right)x_{oi}\,x_{oj} + \frac{|a|}{r_o^3}(\dot{x}_i - \dot{x}_{oi})\,x_{oj}\left[\dot{g} + \frac{r_o}{r}\left(1 - \frac{r_o}{|a|}\right)f_4\right]$$

(60 continued)

$$\frac{\partial \dot{x}_i(t)}{\partial \dot{x}_j(t_K)} \equiv \frac{\partial \dot{x}_i}{\partial \dot{x}_{oj}} = \dot{g}\,\delta_{ij} - \frac{|a|\dot{x}_{oj}}{r^3}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right]\left[-3(t-t_K) + g + \frac{r_o}{|a| n}f_3\right]$$

$$+ \frac{|a|}{r^3}f_2\,x_{oj}\left[x_i + r(\dot{x}_i - \dot{x}_{oi})\frac{\underline{r}\cdot\dot{\underline{r}}}{\mu}\right] + \frac{r_o}{\mu}(\dot{x}_i - \dot{x}_{oi})\,\dot{x}_{oj}\left[\frac{|a|}{r_o}\dot{g} + \frac{|a|}{r}f_4\right]$$

$$+ \frac{r_o}{\mu}(\dot{x}_i - \dot{x}_{oi})\,x_{oi}\dot{f} - \frac{|a|}{\mu}\dot{f}\,x_{oi}\,\dot{x}_{oj}$$

where $i, j = 1, 2, 3$ correspond to the x, y and z components and

$$x_o \equiv x(t_K)$$
$$r_o \equiv r(t_K)$$
$$r \equiv r(t)$$

The inverse, $[\Phi_A(t - t_K)]^{-1}$, can be obtained from the above expression by replacing

$t \to -t$	$r_o \to r$
$\theta \to \theta$	$x \to x_o$
$r \to r_o$	$x_o \to x$

This results in

$f_1 \to -f_1$	$f \to \dot{g}$
$f_2 \to f_2$	$g \to -g$
$f_3 \to -f_3$	$\dot{f} \to -\dot{f}$
$f_4 \to f_4$	$\dot{g} \to f$

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Part I of
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APPROXIMATION OF THE RESTRICTED PROBLEM BY THE TWO-FIXED-CENTER PROBLEM

RAC 720-3

Research Regarding
Guidance and Space Flight Theory
Relative to the Rendezvous Problem
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REPUBLIC AVIATION CORPORATION
Farmingdale, L. I., N. Y.

FOREWORD

This document is Part I of the Second Semiannual Report prepared by Republic Aviation Corporation under NASA Contract No. NAS 8-2605. The report will appear in slightly different format in "Progress Report No. 3 On Studies In The Fields Of Space Flight And Guidance Theory," issued by the Aeroballistics Division of Marshall Space Flight Center.

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CONTENTS

NOTATIONS

A	=	origin of rotating coordinate system
$\underline{A}$	=	position vector from barycenter to center of the rotating system
$\underline{A}_E$	=	the position vector of A relative to the earth
$\underline{E}$	=	position vector of the earth relative to the barycenter at $t=0$
$\underline{E}'$	=	position vector of the earth relative the barycenter, but rotated through an angle ωT
$\Delta\underline{E}$	=	$\underline{E}' - \underline{E}$
J	=	Hamiltonian (Jacobi integral) for the restricted problem
J^*	=	difference between the restricted Hamiltonian and the two-fixed-center Hamiltonian
J_1	=	the part of $\bar{J}$ independent of α, β, and γ
J_2	=	the part of $\bar{J}$ that is a function of α, β, and γ
$\bar{J}$	=	Hamiltonian equivalent to J^* but written in terms of two-fixed-center coordinates and momenta
J^{**}	=	time dependent part of J^*
J'	=	Hamiltonian of two-fixed-center problem
ℓ	=	length of position vector from earth to moon
$\underline{L}$	=	position vector from earth to moon
$\underline{\bar{L}}$	=	position vector of the moon relative to the earth in the rotating system
$\underline{\dot{L}}$	=	velocity of moon with respect to the earth ($\underline{\Omega} \times \underline{L}$)
$\underline{\dot{\bar{L}}}$	=	$\underline{\Omega} \times \underline{\bar{L}}$ in the rotating system
$\underline{P}_A$	=	momentum canonically conjugate to $\underline{\bar{R}}_A$

$\underline{P}'_A$ = momentum canonically conjugate to $\underline{\bar{R}}'_A$

$\underline{R}$ = position vector relative to a point fixed in inertial space e.g. barycenter

$\underline{R}_1$ = position vector relative to the earth

$\underline{R}_2$ = position vector relative to the moon

$\underline{\bar{R}}_A$ = position vector relative to A in the rotating system

$\underline{\bar{R}}_1$ = position vector relative to the earth in the rotating system

$\underline{\bar{R}}_2$ = position vector relative to the moon in the rotating system

$\underline{R}_E$ = position vector from barycenter to earth

$\underline{R}_m$ = position vector from barycenter to moon

$\underline{\bar{R}}'_A$ = position vector relative to A in the rotating system for the two-fixed-center problem

$\underline{R}_A$ = position vector relative to point at A

r_1 = length of position vector relative to earth

r_2 = length of position vector relative to moon

T = a specific period of time

t = time variable

α = constant coefficient of $\underline{L}$ in composition of $\underline{A}$

β = constant coefficient of $\underline{\Omega}$ in composition of $\underline{A}$

γ = constant coefficient of $\dot{\underline{L}}$ in composition of $\underline{A}$

θ = the angle of rotation of the coordinate system about the barycenter after a time T

μ = gravitational constant of the earth

μ' = gravitational constant of the moon

$\underline{\Omega}$ = angular velocity vector of the moon about the earth

ω = the magnitude of angular velocity $\underline{\Omega}$

$grad_{\underline{V}}$ = gradient with respect to the components of $\underline{V}$ taken as coordinates

Subscripts

B = vector relative to the barycenter

0 = initial value

Superscript

dot over quantity = first total time derivative

2 dots over quantity = second total time derivative

REPUBLIC AVIATION CORPORATION
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[redacted]

Approximation of the Restricted Problem by the Two-Fixed Center Problem

By Mary Payne

SUMMARY

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In this report, a perturbation theory of the two-fixed-center problem leading to an approximation for the restricted-three-body problem is developed. It makes use of a generalization of the method developed at MSFC by Schulz-Arenstorff, Davidson, and Sperling.[1] The derivations are carried out in a coordinate system rotating about an accelerated origin, and the generalization consists of the selection of this origin in such a way as to minimize the effects of the non-integrable terms in the perturbation equations. The results of some numerical calculations are presented.

Author

INTRODUCTION

The equations of motion for a vehicle moving in the gravitational fields of the earth and moon are:

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \qquad (1)$$

where $\underline{R}_1$, $\underline{R}_2$, and $\underline{R}$ are the position vectors of the vehicle referred to the earth, the moon, and a point fixed in inertial space, respectively. Lower case letters denote the magnitude of the corresponding vectors. In this report it will be assumed that the earth and moon are moving in circles, under their mutual gravitational attraction, about their common center of mass. This problem is the restricted three-body problem, and the fixed point may be taken to be the center of mass of the earth and the moon. An approximation to the solution of the restricted problem will be sought in terms of the known solution[3] to the Euler problem of two fixed centers of gravitation. The method will, in many respects, follow closely that developed by Schulz-Arenstorff, Davidson, and Sperling.[1] In their procedure, the problem is transformed to a coordinate system rotating about the center of mass. In this rotating system, the Euler problem is taken as the basis of a perturbation theory. Using the initial conditions of the Euler problem as a set of canonical variables, it is shown that[2]

$$\dot{\underline{R}}_0 = -\operatorname{grad}_{\underline{P}_0} J^*$$

and

$$\dot{\underline{P}}_0 = -\operatorname{grad}_{\underline{R}_0} J^* , \qquad (2)$$

where $\underline{R}_0$ is the initial position vector in the rotating system, $\underline{P}_0$ is the momentum vector conjugate to $\underline{R}_0$, and J^* is the difference between the Hamiltonian for the restricted problem (Jacobi integral) and that for the Euler problem, and is given by

$$J^* = \underline{\Omega} \cdot \underline{R}_0 \times \underline{P}_0 - J^{**} . \qquad (3)$$

The solution of the restricted problem is given in terms of an osculating two-fixed center problem with varying initial conditions. If J^{**} were zero, the equations for $\dot{\underline{R}}_0$ and $\dot{\underline{P}}_0$ could be integrated directly. In the Schulz-Arenstorff theory, J^{**} does not vanish and, in fact, contributes appreciably to the variation of $\underline{R}_0$ and $\underline{P}_0$ if the time interval over which the integration extends is too large, or if either the earth or the moon are approached closely by the vehicle during this time interval.

It is the purpose of this report to show that the effect of J^{**} can be reduced by selecting an origin for the rotating system other than the center of mass of the earth and moon. In the course of this development the details of the Schulz-Arenstorff method will be given, and the coordinates for a center of rotation will be determined so that J^{**} and its first time derivative vanish initially.

PRELIMINARY CONSIDERATIONS

Since the two-fixed center problem will be used as the basis of a perturbation theory, it is necessary that the earth and the moon be fixed in the rotating coordinate system. This implies that the origin of this rotating system must be fixed relative to the earth and the moon. The most general of such points will rotate about the barycenter with the angular velocity of the earth and the moon. The radius vector from the barycenter to the origin of the rotating system can be expressed as

$$\underline{A} = \alpha \underline{L} + \beta \underline{\Omega} + \gamma \dot{\underline{L}} , \tag{4}$$

where $\underline{L}$ and $\dot{\underline{L}}$ are the position and velocity vectors, respectively, of the moon relative to the earth in a non-rotating coordinate system, and $\underline{\Omega}$ is the angular velocity of the moon about the earth. From the definition of $\underline{L}$ and $\dot{\underline{L}}$ it is apparent that both vectors are known functions of time. Furthermore, $\underline{L}$ and $\dot{\underline{L}}$ are constant vectors in the rotating system and $\underline{\Omega}$ is constant in both the inertial frame and the rotating system. Thus, the requirement that the point A be fixed relative to the earth and the moon implies that α, β, and γ are numerical constants. The constant β may be chosen arbitrarily, for the point A is used to determine an axis of rotation oriented in the $\underline{\Omega}$ direction, and all points with the same α and γ will lie on the same axis independently of β. Thus, β may be taken as zero without loss of generality, and it will no longer appear in the formulation. Referring to Figure 1, it is seen that $\underline{R}$, $\underline{R}_1$, $\underline{R}_2$, $\underline{L}$, and $\underline{R}_A$, the position vector of the vehicle relative to A, satisfy the following relations:

$$\underline{R}_E = - \frac{\mu'}{\mu + \mu'} \underline{L} \tag{5}$$

$$\underline{R}_M = \frac{\mu}{\mu + \mu'} \underline{L} \tag{6}$$

$$\underline{R}_1 - \underline{R}_2 = \underline{L} \tag{7}$$

$$\underline{R}_A = \underline{R}_1 + \underline{R}_E - \underline{A} = \underline{R}_1 - \left(\alpha + \frac{\mu'}{\mu + \mu'}\right) \underline{L} - \gamma \dot{\underline{L}} \tag{8}$$

$$\underline{R}_A = \underline{R}_2 + \underline{R}_M - \underline{A} = \underline{R}_2 - \left(\alpha - \frac{\mu}{\mu + \mu'}\right) \underline{L} - \gamma \dot{\underline{L}} \tag{9}$$

$$\underline{R} = \underline{A} + \underline{R}_A = \underline{R}_A + \alpha \underline{L} + \gamma \dot{\underline{L}} \tag{10}$$

First, it is necessary to eliminate $\underline{R}$ from Eq. (1) and obtain the equations of motion in terms of $\underline{R}_A$, $\underline{R}_1$, and $\underline{R}_2$. To do this, one may differentiate Eq. (10) twice with respect to time:

$$\ddot{\underline{R}} = \ddot{\underline{R}}_A + \alpha \ddot{\underline{L}} - \gamma \dddot{\underline{L}} \, . \tag{11}$$

Now, the condition that the earth and moon move in circles under their mutual gravitational attraction means that

$$\dot{\underline{L}} = \underline{\Omega} \times \underline{L}$$

and

$$\ddot{\underline{L}} = \underline{\Omega} \times \dot{\underline{L}} = - (\mu + \mu') \frac{\underline{L}}{\ell^3} \, . \tag{12}$$

Differentiation of Eq. (12) (with $\dot{\ell} = 0$, as $\underline{L}$ has constant magnitude), enables us to write Eq. (11) as

$$\ddot{\underline{R}} = \ddot{\underline{R}}_A - \frac{\mu + \mu'}{\ell^3} (\alpha \underline{L} - \gamma \dot{\underline{L}}) \, , \tag{13}$$

and the equations of motion (1) become

$$\ddot{\underline{R}}_A = - \mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} + \frac{\mu + \mu'}{\ell^3} (\alpha \underline{L} - \gamma \dot{\underline{L}}) \, . \tag{14}$$

It should be noted that, at this stage, the coordinate system associated with A is an accelerated system since the origin has uniform circular motion. It is, however, not a rotating system yet - that is, the coordinate axes remain parallel to the inertial axes at the barycenter.

The next step is to transform to rotating coordinates about A. The vectors in this system will be denoted by bars, and the equations of motion become

$$\ddot{\overline{\underline{R}}}_A = - \mu \frac{\overline{\underline{R}}_1}{r_1^3} - \mu' \frac{\overline{\underline{R}}_2}{r_2^3} + \frac{\mu + \mu'}{\ell^3} (\alpha \overline{\underline{L}} + \gamma \dot{\overline{\underline{L}}}) - 2 \underline{\Omega} \times \dot{\overline{\underline{R}}}_A - \underline{\Omega} \times \left(\underline{\Omega} \times \overline{\underline{R}}_A \right) . \tag{15}$$

It should be noted that, in this rotating coordinate system, the earth and the moon are fixed, with position vector $\overline{\underline{L}}$ of the moon relative to the earth as a constant vector. The vector $\dot{\underline{L}}$ does not represent the velocity of the moon (which is zero), but is a vector mutually perpendicular to $\bar{\underline{L}}$ and $\underline{\Omega}$, and satisfying Eq. (16) with bars over the vectors. As the rotating system has angular velocity $\underline{\Omega}$, it follows of course, that $\bar{\underline{\Omega}}$ and $\underline{\Omega}$ are identical.

A constant of motion for the problem in the rotating system may now be obtained by dotting Eq. (15) with $\dot{\bar{\underline{R}}}_A$ and noting that the earth and the moon are fixed in this system, so that

$$\dot{\bar{\underline{R}}}_A = \dot{\bar{\underline{R}}}_1 = \dot{\bar{\underline{R}}}_2 \,. \tag{16}$$

Thus,

$$\dot{\bar{\underline{R}}}_A \cdot \ddot{\bar{\underline{R}}}_A = \frac{d}{dt}\left(\frac{\dot{\bar{\underline{R}}}_A^{\,2}}{2}\right) = -\mu \frac{\bar{\underline{R}}_1 \cdot \dot{\bar{\underline{R}}}_1}{r_1^{\,3}} - \mu' \frac{\bar{\underline{R}}_2 \cdot \dot{\bar{\underline{R}}}_2}{r_2^{\,3}} \tag{17}$$

$$- \frac{\mu - \mu'}{\lambda^3}\left(\alpha \dot{\bar{\underline{R}}}_A \cdot \bar{\underline{L}} + \gamma \dot{\bar{\underline{R}}}_A \cdot \dot{\bar{\underline{L}}}\right) + \left(\underline{\Omega} \times \dot{\bar{\underline{R}}}_A\right) \cdot \left(\underline{\Omega} \times \bar{\underline{R}}_A\right)$$

$$= \frac{d}{dt}\left[\frac{\mu}{r_1} + \frac{\mu'}{r_2} + \frac{\mu - \mu'}{\lambda^3}\left(\alpha \bar{\underline{R}}_A \cdot \bar{\underline{L}} + \gamma \bar{\underline{R}}_A \cdot \dot{\bar{\underline{L}}}\right) + \tfrac{1}{2}\left(\underline{\Omega} \times \bar{\underline{R}}_A\right)^2\right]$$

as $\bar{\underline{L}}$ and $\dot{\bar{\underline{L}}}$ are constant vectors. Denoting the constant of motion by J:

$$J = \tfrac{1}{2}\dot{\bar{\underline{R}}}_A^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \frac{\mu + \mu'}{\lambda^3}\left(\alpha \bar{\underline{R}}_A \cdot \bar{\underline{L}} + \gamma \bar{\underline{R}}_A \cdot \dot{\bar{\underline{L}}}\right) - \tfrac{1}{2}\left(\underline{\Omega} \times \bar{\underline{R}}_A\right)^2 . \tag{18}$$

It may now be shown that, if the vector

$$\underline{P}_A = \dot{\bar{\underline{R}}}_A + \underline{\Omega} \times \bar{\underline{R}}_A \tag{19}$$

is regarded as the momentum conjugate to $\bar{\underline{R}}_A$, the integral J of the motion becomes the Hamiltonian. To prove this, substitute for $\dot{\bar{\underline{R}}}_A$, using Eq. (19), in Eq. (18):

$$J = \tfrac{1}{2}\left(\underline{P}_A - \underline{\Omega} \times \bar{\underline{R}}_A\right)^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \frac{\mu + \mu'}{\lambda^3}\left(\alpha \bar{\underline{R}}_A \cdot \bar{\underline{L}} + \gamma \bar{\underline{R}}_A \cdot \dot{\bar{\underline{L}}}\right) - \tfrac{1}{2}\left(\underline{\Omega} \times \bar{\underline{R}}_A\right)^2 \tag{20}$$

$$= \tfrac{1}{2}\underline{P}_A^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \underline{\Omega} \cdot \bar{\underline{R}}_A \times \underline{P}_A - \frac{\mu - \mu'}{\lambda^3}\left(\alpha \bar{\underline{R}}_A \cdot \bar{\underline{L}} + \gamma \bar{\underline{R}}_A \cdot \dot{\bar{\underline{L}}}\right).$$

If $\underline{\bar{R}}_A$ and $\underline{P}_A$ are conjugate vectors, Hamilton's equations,

$$\dot{\underline{\bar{R}}}_A = \text{grad}_{\underline{P}_A} J = \underline{P}_A - \underline{\Omega} \times \underline{\bar{R}}_A , \tag{21}$$

and

$$\dot{\underline{P}}_A = - \text{grad}_{\underline{\bar{R}}_A} J = \text{grad}_{\underline{\bar{R}}_A} \left(\frac{\mu}{r_1} + \frac{\mu'}{r_2} \right) - \underline{\Omega} \times \underline{P}_A + \frac{\mu + \mu'}{r^3} \left(\alpha \underline{\bar{L}} + \gamma \dot{\underline{\bar{L}}} \right) \tag{22}$$

must be satisfied. It is evident that Eq. (21) is identical with Eq. (19), defining the relation between velocity $\dot{\underline{\bar{R}}}_A$ and the momentum $\underline{P}_A$ conjugate to $\underline{\bar{R}}_A$. Now, it will be shown that Eq. (22) reduces to the equations of motion (15) in the rotating system. First,

$$\text{grad}_{\underline{\bar{R}}_A} \frac{\mu}{r_1} = - \frac{\mu}{r_1^{\,2}} \text{grad}_{\underline{\bar{R}}_A} r_1 . \tag{23}$$

But,

$$r_1^{\,2} = \underline{\bar{R}}_1 \cdot \underline{\bar{R}}_1 ; \tag{24}$$

hence,

$$\begin{aligned} 2\, r_1 \,\text{grad}_{\underline{\bar{R}}_A} r_1 &= \text{grad}_{\underline{\bar{R}}_A} r_1^{\,2} \\ &= \text{grad}_{\underline{\bar{R}}_A} \left(\underline{\bar{R}}_A - \underline{\bar{R}}_E + \underline{\bar{A}} \right)^2 \\ &= \text{grad}_{\underline{\bar{R}}_A} \left[\underline{\bar{R}}_A^{\,2} + 2\, \underline{\bar{R}}_A \cdot \left(\underline{\bar{A}} - \underline{\bar{R}}_E \right) + \left(\underline{\bar{A}} - \underline{\bar{R}}_E \right)^2 \right] \\ &= 2 \underline{\bar{R}}_A - 2 \left(\underline{\bar{A}} - \underline{\bar{R}}_E \right) = 2\, \underline{\bar{R}}_1 , \end{aligned} \tag{25}$$

so that, finally,

$$\text{grad}_{\underline{\bar{R}}_A} r_1 = \frac{\underline{\bar{R}}_1}{r_1}$$

and

$$\text{grad}_{\underline{\bar{R}}_A} \frac{\mu}{r_1} = - \frac{\mu \underline{\bar{R}}_1}{r_1^{\,3}} . \tag{26}$$

Similarly,

$$\operatorname{grad}_{\underline{\overline{R}}_A} \frac{\mu}{r_2} = - \frac{\mu \underline{\overline{R}}_2}{r_2^{\,3}} , \tag{27}$$

so that Eq. (22) may be written as

$$\dot{\underline{P}}_A = - \frac{\mu \underline{\overline{R}}_1}{r_1^{\,3}} - \frac{\mu' \underline{\overline{R}}_2}{r_2^{\,3}} - \underline{\Omega} \times \underline{P}_A + \frac{\mu - \mu'}{c^3} \left(\alpha \, \underline{\overline{L}} + \gamma \, \dot{\underline{\overline{L}}} \right) . \tag{28}$$

Now, from Eq. (19),

$$\dot{\underline{P}}_A = \ddot{\underline{\overline{R}}}_A - \underline{\Omega} \times \dot{\underline{\overline{R}}}_A , \tag{29}$$

and use of this relation for $\dot{\underline{P}}_A$ and Eq. (19) for $\underline{P}_A$ in Eq. (28) yields

$$\ddot{\underline{\overline{R}}}_A + \underline{\Omega} \times \dot{\underline{\overline{R}}}_A = - \frac{\mu \underline{\overline{R}}_1}{r_1^{\,3}} - \frac{\mu' \underline{\overline{R}}_2}{r_2^{\,3}} - \underline{\Omega} \times \dot{\underline{\overline{R}}}_A - \underline{\Omega} \times \left(\underline{\Omega} \times \underline{\overline{R}}_A \right) - \frac{\mu - \mu'}{c^3} \left(\alpha \, \underline{\overline{L}} - \gamma \, \dot{\underline{\overline{L}}} \right) . \tag{30}$$

Finally, if the $\underline{\Omega} \times \dot{\underline{\overline{R}}}_A$ on the left is transposed to the right hand side of Eq. (30), it becomes identical with the equations of motion (15) in the rotating system.

RELATION BETWEEN THE TWO-FIXED CENTER PROBLEM AND THE RESTRICTED PROBLEM

A Hamiltonian, J, has now been obtained for the restricted problem in a rotating coordinate system with the origin at A:

$$J = \tfrac{1}{2} \underline{P}_A^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \underline{\Omega} \cdot \underline{\overline{R}}_A \times \underline{P}_A - \frac{\mu - \mu'}{c^3} \left(\alpha \, \underline{\overline{R}}_A \cdot \underline{\overline{L}} + \gamma \, \underline{\overline{R}}_A \cdot \dot{\underline{\overline{L}}} \right) , \tag{31}$$

with

$$\underline{\overline{A}} = \alpha \, \underline{\overline{L}} + \gamma \, \dot{\underline{\overline{L}}} , \tag{32}$$

referred to the barycenter of earth and moon, and

$$\underline{P}_A = \dot{\underline{\bar{R}}}_A + \underline{\Omega} \times \underline{\bar{R}}_A . \tag{33}$$

The development so far differs slightly from that of Schulz-Arenstorff, Davidson, and Sperling[1] in two respects: it has been carried out in three dimensions instead of two, and the center of the rotating coordinate system is at A instead of the barycenter. Following their development, a solution of Eq. (31) in terms of the solution of the two-fixed center problem is now sought. For the two-fixed center problem, the Hamiltonian is given by:

$$J' = \frac{1}{2} \underline{P}_A{}'^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} , \tag{34}$$

and the Hamilton equations are

$$\dot{\underline{\bar{R}}}_A{}' = \mathrm{grad}_{\underline{P}_A{}'} J' = \underline{P}_A{}'$$

and

$$\dot{\underline{P}}_A{}' = - \mathrm{grad}_{\underline{\bar{R}}_A} J' = - \mu \frac{\bar{R}_1{}'}{r_1^3} - \mu' \frac{\bar{R}_2{}'}{r_2^3} . \tag{35}$$

Denoting the solution of the two-fixed center problem by primes and that for the restricted problem without primes, the solution sought is to have the form

$$\underline{R}\left(\underline{R}_0, \underline{P}_0, t\right) = \underline{R}'\left(\underline{R}_0(t), \underline{P}_0(t), t\right)$$

and (36)

$$\underline{P}\left(\underline{R}_0, \underline{P}_0, t\right) = \underline{P}'\left(\underline{R}_0(t), \underline{P}_0(t), t\right) .$$

Thus, the problem is reduced to finding the time dependence of the initial conditions in the solution of the two-fixed center problem that provide the solution of the restricted problem in the same functional form as that of the two-fixed center solution.

The theorem, mentioned in the introduction, on the equations determining the time variation of the initial conditions will now be given a precise statement.

Theorem: If $\underline{R}(\underline{R}_0, \underline{P}_0, t)$ and $\underline{P}(\underline{R}_0, \underline{P}_0, t)$ constitute the solution of a problem with Hamiltonian $J(\underline{R}, \underline{P})$ while $\underline{R}'(\underline{R}_0, \underline{P}_0, t)$ and $\underline{P}'(\underline{R}_0, \underline{P}_0, t)$ constitute the solution of a problem with Hamiltonian $J'(R', P')$ with

$$\underline{R}(\underline{R}_0, \underline{P}_0, 0) = \underline{R}'(\underline{R}_0, \underline{P}_0, 0) = \underline{R}_0$$

and

$$\underline{P}(\underline{R}_0, \underline{P}_0, t) = \underline{P}'(\underline{R}_0, \underline{P}_0, 0) = \underline{P}_0 \tag{37}$$

then Eqs. (36) are satisfied with $\underline{R}_0(t)$ and $\underline{P}_0(t)$, determined by the equations

$$\dot{\underline{R}}_0(t) = \text{grad}_{\underline{P}_0} J^*(\underline{R}_0, \underline{P}_0, t)$$

and

$$\dot{\underline{P}}_0(t) = -\text{grad}_{\underline{R}_0} J^*(\underline{R}_0, \underline{P}_0, t), \tag{38}$$

where

$$\bar{J}(\underline{R}', \underline{P}') = J(\underline{R}', \underline{P}') - J'(\underline{R}', \underline{P}') = J^*(\underline{R}_0, \underline{P}_0, t) \tag{39}$$

Wherever $\underline{R}_0$ and $\underline{P}_0$ occur on the right hand side as a result of the gradient operations, they are to be replaced by $\underline{R}_0(t)$ and $\underline{P}_0(t)$, respectively.

This theorem has been proven by Arenstorf[2] in an unpublished note and will now be applied.

To obtain the differential equations for $\underline{R}_0(t)$ and $\underline{P}_0(t)$, $\bar{J}$ must be written in terms of $\bar{\underline{R}}'_A$ and $\underline{P}'_A$, associated with the two-fixed outer problem. That is,

$$\begin{aligned}\bar{J} &= J(\bar{\underline{R}}'_A, \underline{P}'_A) - J'(\bar{\underline{R}}'_A, \underline{P}'_A) \\ &= -\underline{\Omega} \cdot \bar{\underline{R}}'_A \times \underline{P}'_A - \frac{\mu + \mu'}{r^3}(\alpha \bar{\underline{R}}'_A \cdot \underline{L} + \gamma \bar{\underline{R}}'_A \cdot \underline{L}'),\end{aligned} \tag{40}$$

where $J(\bar{\underline{R}}'_A, \underline{P}'_A)$ is obtained from Eq. (31) by replacing $\bar{\underline{R}}_A$ and $\underline{P}_A$ by the corresponding primed quantities, and $J'(\bar{\underline{R}}'_A, \underline{P}'_A)$ is given by Eq. (34).

It is now necessary to obtain J^* by expressing $\bar{J}$ in terms of the initial conditions of the two-fixed center problem. This is very difficult to do exactly, as the solution[3] of the two-fixed center problem is given in terms of elliptic functions with the initial conditions entering not only in coefficients of these functions but also in their moduli. Therefore, the solution of the two-fixed center problem is a transcendental function of the initial conditions. An approximate solution is, however, obtainable by expanding $\bar{J}$ as a power series in time:

$$
\begin{aligned}
\bar{J} &= \bar{J}(0) + \dot{\bar{J}}(0)\,t + \ddot{\bar{J}}(0)\,\frac{t^2}{2} + \dots \\
&= J^*(0) + \dot{J}^*(0)\,t + \ddot{J}^*(0)\,\frac{t^2}{2} + \dots \, .
\end{aligned}
\tag{41}
$$

Using Eq. (40), the first time derivative of $\bar{J}$ is

$$
\dot{\bar{J}} = -\underline{\Omega}\cdot \dot{\underline{\bar{R}}}'_A \times \underline{P}'_A - \underline{\Omega}\cdot \underline{\bar{R}}'_A \times \dot{\underline{P}}'_A - \frac{\mu-\mu'}{\lambda^3}\left(\alpha\dot{\underline{\bar{R}}}_A\cdot \underline{L} + \gamma \dot{\underline{\bar{R}}}'_A\cdot \dot{\bar{\underline{L}}}\right). \tag{42}
$$

Now, Eq. (42) contains time derivatives of $\underline{\bar{R}}_A$ and $\underline{P}_A$, which may be eliminated by means of the Hamilton equations (35) for the two-fixed center problem:

$$
\dot{\bar{J}} = -\underline{\Omega}\cdot \underline{P}'_A \times \underline{P}'_A - \underline{\Omega}\cdot \underline{\bar{R}}'\times\left(-\frac{\mu\bar{R}'_1}{r_1^3} - \frac{\mu'\bar{R}_2}{r_2^3}\right) - \frac{\mu+\mu'}{\lambda^3}\underline{P}'_A\cdot\left(\alpha\underline{L} + \gamma\dot{\bar{\underline{L}}}\right). \tag{43}
$$

The first term in this equation vanishes. Evaluation of $\bar{J}$ and $\dot{\bar{J}}$ at $t=0$ yields

$$
\bar{J}(0) = J^*(0) = -\underline{\Omega}\cdot \underline{\bar{R}}'_{A0}\times \underline{P}'_{A0} - \frac{\mu-\mu'}{\lambda^3}\underline{\bar{R}}'_{A0}\cdot\left(\alpha\underline{L} + \gamma\dot{\bar{\underline{L}}}\right). \tag{44}
$$

and

$$
\dot{\bar{J}}(0) = \dot{J}^*(0) = -\underline{\Omega}\cdot \underline{\bar{R}}'_{A0}\times\left(\frac{-\mu\bar{R}'_{10}}{r_{10}^3} - \frac{\mu'\bar{R}'_{20}}{r_{20}^3}\right) - \frac{\mu-\mu'}{\lambda^3}\underline{P}'_{A0}\cdot\left(\alpha\underline{L} + \gamma\dot{\bar{\underline{L}}}\right) \tag{45}
$$

Setting

$$
J_1 = -\underline{\Omega}\cdot \underline{\bar{R}}'_{A0}\times \underline{P}'_{A0} \tag{46}
$$

and

$$
\begin{aligned}
J_2 = -\frac{\mu-\mu'}{\lambda^3}\underline{\bar{R}}'_{A0}\cdot\left(\alpha\underline{L} - \gamma\dot{\bar{\underline{L}}}\right) + t\,\underline{\Omega}\cdot\underline{\bar{R}}'_{A0}\times\left(\frac{\mu\bar{R}'_{10}}{r_{10}^3} + \frac{\mu'\bar{R}'_{20}}{r_{20}^3}\right) \\
- \frac{\mu+\mu'}{\lambda^3}\underline{P}'_{A0}\cdot\left(\alpha\underline{L} + \gamma\dot{\bar{\underline{L}}}\right) + \dots ,
\end{aligned}
\tag{47}
$$

so that

$$
J^* = J_1 + J_2 \, .
$$

Application of the Arenstorf theorem, now yields

$$\dot{\underline{\bar{R}}}'_{A0} = -\operatorname{grad}_{\underline{P}'_{A0}} J^* = -\underline{\Omega} \times \underline{\bar{R}}'_{A0} - \operatorname{grad}_{\underline{P}'_{A0}} J_2 \tag{48}$$

and

$$\dot{\underline{P}}'_{A0} = -\operatorname{grad}_{\underline{\bar{R}}'_{A0}} J^* = -\underline{\Omega} \times \underline{P}'_{A0} - \operatorname{grad}_{\underline{\bar{R}}'_{A0}} J_2 \tag{49}$$

as the differential equations for the variation of the two-fixed center initial conditions, which must be included in the two-fixed center solution in order that it may become the solution of the restricted problem.

If J_2 were zero, Eqs. (48) and (49) would integrate immediately. They would simply say that $\underline{\bar{R}}_{A0}$ and $\underline{P}_{A0}$ rotate clockwise with angular velocity $\underline{\Omega}$. That is, in the rotating system the solution of the restricted problem at time T would be given by the solution of the two-fixed center problem at time T, with initial conditions obtained from those of the restricted problem by a clockwise rotation through ΩT about the point A. For T=0, the restricted and two-fixed center problems have the same initial conditions and, hence, have exactly the same solution.

Actually, of course, J_2 does not vanish, and it is here that the selection of the point A enters. Every term of J_2 involves either $\underline{\bar{R}}_{A0}$ or $\underline{P}_{A0}$, which depend on the selection of the point A, so that this point should be selected so as to minimize the contribution of J_2 to the variation of the initial conditions. This could be done in various ways. Inasmuch as the position of the point A depends on the two parameters α and γ, it is evident that only two conditions can be imposed on the selection of A. Several such conditions suggest themselves immediately:

(1) Determine α and γ so that in J_2 the constant term and the coefficient of t vanish for the initial values of $\underline{\bar{R}}'_{A0}$ and $\underline{P}'_{A0}$.

(2) Determine α and γ so that J_2 vanish for t=0, with initial values of $\underline{\bar{R}}'_{A0}$ and $\underline{P}'_{A0}$, and also vanish at t=T, with the rotated values of $\underline{\bar{R}}'_{A0}$ and $\underline{P}'_{A0}$ determined by J_1 at time T.

(3) Determine α and γ so that the square of J_2 is minimized over the time interval 0 to T, using either the initial values of $\underline{\bar{R}}'_{A0}$ and $\underline{P}'_{A0}$ or their time dependent values determined by J_1 over the interval.

The first method has the disadvantage that the validity of the approximation would deteriorate with time, and there is no obvious way of estimating the duration of validity. The other two methods have the disadvantage that, if the time interval specified is too long, the approximation would not be valid, even initially, and again, a criterion for "too long" is missing. It was, therefore, decided to try the first method, which would give some insight into the duration of validity, and might very well produce results of practical value.

DETERMINATION OF α AND γ

In accordance with the conclusion of the last section, α and γ are to be determined by the equations

$$\underline{\bar{R}}_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \underline{\dot{\bar{L}}}) = 0 \tag{50}$$

and

$$\underline{\Omega} \cdot \underline{\bar{R}}_{A0} \times \left(\frac{\mu \underline{\bar{R}}_{10}}{r_{10}^3} + \mu' \frac{\underline{\bar{R}}_{20}}{r_{20}^3} \right) - \frac{\mu + \mu'}{\lambda^3} \underline{P}_{A0} \cdot (\alpha \underline{\bar{L}} + \gamma \underline{\dot{\bar{L}}}) = 0, \tag{51}$$

so that the first two terms in the power series expansion of J_2 in Eq. (47) vanish. The primes have been omitted in Eqs. (50) and (51) because the initial values of $\underline{\bar{R}}'_{A0}$ and $\underline{P}'_{A0}$, regarded as variable parameters for the restricted problem, are the initial values of the restricted problem by the Arenstorf theorem.[2] Now, $\underline{\bar{R}}_{A0}$ and $\underline{P}_{A0}$ depend on the selection of the point A, so that, for the determination of α and γ from Eqs. (50) and (51), they should be replaced by the position and momentum of the vehicle relative to some point independent of A. A particularly compact form is obtained for the equations of α and γ by replacing $\underline{P}_{A0}$ by $\underline{P}_{10}$ and $\underline{\bar{R}}_{A0}$ by $\underline{\bar{R}}_{10}$ or $\underline{\bar{R}}_{20}$, as follows. First, since from Eq. (19)

$$\underline{P}_{A0} = \underline{\dot{\bar{R}}}_{A0} + \underline{\Omega} \times \underline{\bar{R}}_{A0}, \tag{52}$$

for any point A fixed relative to earth and moon, it follows that

$$\underline{P}_{10} = \underline{\dot{\bar{R}}}_{10} + \underline{\Omega} \times \underline{\bar{R}}_{10}. \tag{53}$$

Therefore, since in the rotating system the velocity of the vehicle relative to the earth is the same as that relative to A (both are fixed points in the rotating system),

$$\begin{aligned} \underline{P}_{A0} &= \underline{P}_{10} + \underline{\Omega} \times (\underline{\bar{R}}_{A0} - \underline{\bar{R}}_{10}) \\ &= \underline{P}_{10} - \underline{\Omega} \times \left(\left(\alpha + \frac{\mu'}{\mu + \mu'}\right) \underline{\bar{L}} + \gamma \underline{\dot{\bar{L}}} \right) \\ &= \underline{P}_{10} - \left(\alpha + \frac{\mu'}{\mu + \mu'}\right) \underline{\dot{\bar{L}}} + \gamma \frac{\mu + \mu'}{\lambda^3} \underline{\bar{L}}, \end{aligned} \tag{54}$$

on making use of Eqs. (8) and (12). Thus, the third term of Eq. (51) will be proportional to

$$\underline{P}_{A0} \cdot (\alpha \underline{L} + \gamma \dot{\underline{L}}) = \underline{P}_{10} \cdot (\alpha \underline{L} + \gamma \dot{\underline{L}}) - \gamma \frac{\mu'}{\lambda} , \tag{55}$$

where the terms in $\alpha\gamma$ have canceled out.

Again using Eq. (8), the first term of Eq. (51) will involve

$$\begin{aligned} \underline{\Omega} \cdot \bar{R}_{A0} \times \bar{R}_{10} &= - \underline{\Omega} \times \left[(\alpha + \frac{\mu'}{\mu + \mu'}) \underline{L} + \gamma \dot{\underline{L}} \right] \cdot \bar{R}_{10} \\ &= - \bar{R}_{10} \cdot \left[(\alpha + \frac{\mu'}{\mu + \mu'}) \dot{\underline{L}} - \gamma \frac{\mu + \mu'}{\lambda^3} \underline{L} \right] , \end{aligned} \tag{56}$$

and the second term will be proportional to

$$\begin{aligned} \underline{\Omega} \cdot \bar{R}_{A0} \times \bar{R}_{20} &= - \underline{\Omega} \times \left[(\alpha - \frac{\mu}{\mu + \mu'}) \underline{L} + \gamma \dot{\underline{L}} \right] \cdot \bar{R}_{20} \\ &= - \bar{R}_{20} \cdot \left[(\alpha - \frac{\mu}{\mu + \mu'}) \dot{\underline{L}} - \gamma \frac{\mu + \mu'}{\lambda^3} \underline{L} \right] , \end{aligned} \tag{57}$$

so that Eq. (51) may now be written as follows:

$$\begin{aligned} &\frac{\mu}{r_{10}^3} \left[- (\alpha + \frac{\mu'}{\mu + \mu'}) \bar{R}_{10} \cdot \dot{\underline{L}} + \gamma \frac{\mu + \mu'}{\lambda^3} \bar{R}_{10} \cdot \underline{L} \right] \\ &+ \frac{\mu'}{r_{20}^3} \left[- (\alpha - \frac{\mu}{\mu + \mu'}) \bar{R}_{20} \cdot \dot{\underline{L}} + \gamma \frac{\mu + \mu'}{\lambda^3} \bar{R}_{20} \cdot \underline{L} \right] \\ &- \frac{\mu + \mu'}{\lambda^3} \left[\underline{P}_{10} \cdot (\alpha \underline{L} + \gamma \dot{\underline{L}}) - \frac{\mu'}{\lambda} \gamma \right] = 0 \end{aligned} \tag{58}$$

or, collecting terms in α and γ:

$$
\begin{aligned}
& -\alpha\left[\underline{\bar{R}}_{10}\cdot\underline{\dot{\bar{L}}}\left(\frac{\mu'}{r_{10}^{3}}+\frac{\mu'}{r_{20}^{3}}\right)+\frac{\mu+\mu'}{\ell^{3}}\,\underline{P}_{10}\cdot\underline{\dot{\bar{L}}}\right] \\
& +\frac{\mu+\mu'}{\ell^{3}}\,\gamma\left[\mu\,\frac{\underline{\bar{R}}_{10}\cdot\underline{\dot{\bar{L}}}}{r_{10}^{3}}+\mu'\,\frac{\underline{\bar{R}}_{20}\cdot\underline{\bar{L}}}{r_{20}^{3}}+\frac{\mu'}{\ell}-\underline{\dot{P}}_{10}\cdot\underline{\dot{\bar{L}}}\right] \\
& -\frac{\mu\,\mu'}{\mu+\mu'}\,\underline{\bar{R}}_{10}\cdot\underline{\dot{\bar{L}}}\left(\frac{1}{r_{10}^{3}}-\frac{1}{r_{20}^{3}}\right) \\
& =0
\end{aligned}
\tag{58}
$$

where use has been made of the fact that

$$
\underline{\bar{R}}_{10}\cdot\underline{\dot{\bar{L}}}=\underline{\bar{R}}_{20}\cdot\underline{\dot{\bar{L}}}\quad . \tag{59}
$$

Using Eq. (8) once more, one obtains for Eq. (50):

$$
\begin{aligned}
& \underline{\bar{R}}_{10}\cdot(\alpha\,\underline{L}+\gamma\,\underline{\dot{\bar{L}}})-\left[(\alpha-\frac{\mu'}{\mu+\mu'})\underline{\bar{L}}+\gamma\,\underline{\dot{\bar{L}}}\right]\cdot\left[\alpha\,\underline{\bar{L}}+\gamma\,\underline{\dot{\bar{L}}}\right] \quad . \\
& =\underline{\bar{R}}_{10}\cdot(\alpha\,\underline{\bar{L}}+\gamma\,\underline{\dot{\bar{L}}})-\alpha\,(\alpha-\frac{\mu'}{\mu+\mu'})\,\ell^{2}-\gamma^{2}\cdot\dot{\ell}^{2} \\
& =-\alpha^{2}\ell^{2}+\alpha\,(\underline{\bar{R}}_{10}\cdot\underline{\bar{L}}-\frac{\mu'}{\mu+\mu'}\,\ell^{2})-\gamma^{2}\frac{\mu+\mu'}{\ell}+\gamma\,(\underline{\bar{R}}_{10}\cdot\underline{\dot{\bar{L}}})=0
\end{aligned}
\tag{60}
$$

If Eqs. (58) and (60) are solved for α and γ, a point A is determined so that the following procedure should give an approximation to the restricted problem valid for a time interval whose length depends on the size of J^* and the rate of variation of $\underline{\bar{R}}'_{10}$ and $\underline{P}'_{10}$. The procedure is carried out in the rotating system as follows:

> Modify the initial conditions of the restricted problem by a clockwise rotation through ω T about the point A, and solve the two-fixed center problem with these modified initial conditions. Then, $\underline{\bar{R}}'_{A}(T)$ and $\underline{P}'_{A}(T)$, given by the two-fixed center problem, should match $\underline{\bar{R}}_{A}(T)$ given by the restricted problem with unmodified initial conditions.

APPLICATION OF THE METHOD

In order to carry out a numerical test of the method, use was made of the Republic interplanetary trajectory program. The input for this program requires that initial conditions be given in a coordinate system with its origin at the earth and axes with fixed directions in space. The z-axis points towards the pole star, the x-axis points to the first point of Aries, and the y-axis is selected so that the system is orthogonal and right-handed. The output includes coordinates and velocities of the vehicle in this same system. An option is available which fixes the moon at any desired point on its orbit and computes a two-fixed center problem for this fixed position of the moon and given initial conditions. A set of initial conditions is available which yields a lunar trajectory (referred to, henceforth, as the base case) with a moving moon, starting near the earth, closely circling the moon and returning to the earth. Thus, to test the application one could modify the coordinates and velocities at various points on this base case and compute a two-fixed center problem from the modified conditions to obtain a comparison, which should indicate the time intervals over which the approximation is useful for various portions of the trajectory.

The modification of the initial conditions derived in the preceding sections was carried out in a rotating system, and it is now necessary to transform this modification for use in the coordinate system of the Interplanetary program. To see how this may be done, suppose for the moment that the point A is at the barycenter, i.e., α and γ are both zero, and that the fixed and rotating systems are coincident at $t = 0$. It is evident, in this case, that the two-fixed center orbit obtained from the initial conditions, modified by a clockwise rotation through an angle θ about the barycenter, is exactly the same relative to the earth and moon as if the initial conditions had been unmodified and the earth and moon had been rotated counterclockwise through θ about the barycenter. Now, the angle θ is ωT, where T is the time at which the comparison is to be made. Hence, if the earth, the moon, and the two-fixed center orbit, corresponding to the modified initial conditions, is rigidly rotated counterclockwise through ωT, the earth and moon will coincide with their positions at time T in the fixed system, and the point corresponding to time T on the two-fixed center orbit is the one to be compared with the restricted problem carried out in the fixed system. Moreover, this counterclockwise rotation just transforms the two-fixed center problem, with modified initial conditions and earth and moon in initial position, into that with unmodified initial conditions and earth and moon in their T positions. Therefore, for α and γ both zero, the comparison can be made, using the interplanetary program by fixing the moon in its T position and referring the unmodified initial conditions to the coordinate system centered at the earth at time T. This is indicated in Fig. 2, where the unprimed initial conditions are referred to the earth at $t = 0$, and the primed initial conditions refer to the earth at $t = T$. The initial conditions are fixed.

A comment on the relation between the momentum vector $\underline{P}_B$, conjugate to $\overline{\underline{R}}_B$, and the velocity vector $\dot{\overline{\underline{R}}}_B$, where B is used to indicate that the barycenter is the origin of the rotating system, is now in order. Recalling the definition of P_A in Eq. (19), it follows that

$$\underline{P}_B = \dot{\overline{\underline{R}}}_B - \underline{\Omega} \times \overline{\underline{R}}_B \,, \tag{61}$$

and hence $\underline{P}_B$ is simply the velocity vector in the fixed system with its components referred to the instantaneous rotating axes. Since it has been assumed that the fixed and rotating systems are coincident at t = 0, it follows that

$$\underline{P}_{B_0} = \dot{\underline{R}}_{B_0} \,, \tag{62}$$

where $\dot{\underline{R}}_B$ is in the fixed system (recall that bars denote rotating system). At time T, if the $\underline{P}_B$ vector is rotated through ωT counterclockwise, it will become the $\dot{\underline{R}}_B$ vector. But this is just the transformation that has been used to translate the two-fixed center approximation from the rotating to the fixed system.

Thus, if the barycenter is the origin of the rotating system (i.e., $\alpha = \gamma = 0$), the prescription for the approximation is the following:

(1) Let

$$\underline{\Delta E} = \underline{E}' - \underline{E} = \frac{-\mu'}{\mu + \mu'} \left(\underline{L}(T) - \underline{L}(0) \right) \tag{63}$$

be the displacement of the earth in time T.

(2) Set

$$\underline{R}'_{10} = \underline{R}_{10} - \underline{\Delta E} = \underline{R}_{10} - \frac{\mu'}{\mu + \mu'} \Delta \underline{L} \tag{64}$$

and

$$\dot{\underline{R}}'_{10} = \dot{\underline{R}}_{10} \,, \tag{65}$$

since a translation of the origin will not affect the velocity.

(3) Fix the moon at $\underline{L}(T)$, that is in its position at time T relative to the earth.

(4) Solve the two-fixed center problem with the moon (fixed at $\underline{L}(T)$) and initial conditions $\underline{R}'_{10}$ and $\dot{\underline{R}}_{10}$ to obtain an approximation at time T to the restricted problem with initial conditions $\underline{R}_{10}$ and $\dot{\underline{R}}_{10}$ and moon initially at $\underline{L}(0)$.

The analysis for a system rotating about any point other than the barycenter is carried out in a similar way, but the algebra is more complicated. The origin of the rotating system is to be the point A, defined by Eq. (4), with α and γ determined from Eqs. (58) and (60).

In Fig. 3, the vector A and the original and modified initial conditions are shown in the rotating system.

Again, it is seen that the two-fixed center problem, with primed initial conditions and unprimed positions of earth and moon, is related to that with unprimed initial conditions and primed positions of earth and moon by a rigid rotation which is the rotation part of the transformation carrying the rotating system into the fixed system. It must be remembered, however, that unlike the barycenter B, which may be regarded as a fixed inertial point, A is an accelerated point in inertial space, so that more than a rotation is required to transform back from the rotating system to the fixed system. In Fig. 4, the system rotating about A is shown at $t = 0$ and $t = T$.

It is now easy to see that the translation required to complete the transformation to axes moving with A, but with fixed directions, is a translation from A to A′. Actually, this translation need not be considered further because it is desired to find modification in the initial conditions relative to the earth rather than relative to A.

Referring again to Fig. 3, it is seen that the primed positions of the earth and the moon define a line parallel to that of the earth and moon at time T in the fixed system. Thus, just as in the barycenter case,

$$\underline{R}_{10}' = \underline{R}_{10} - \underline{\Delta E}$$

and (66)

$$\underline{P}' = \underline{P}.$$

To obtain $\underline{\Delta E}$, one may note that $\underline{\Delta E}$ is obtained by a rotation of E through ωT about A and that this $\underline{\Delta E}$ is just the negative of a rotation of A through ωT about E. The vector $\underline{A}$, relative to E, is given by

$$\underline{A}_E = \underline{A} + \frac{\mu'}{\mu + \mu'} \underline{L} = (\alpha + \frac{\mu'}{\mu + \mu'}) \underline{L} + \gamma \dot{\underline{L}}, \qquad (67)$$

and the change in $\underline{A}_E$ induced by a rotation of $\underline{A}_E$ through ωT about E is given by

$$\underline{\Delta A}_E = (\alpha + \frac{\mu'}{\mu + \mu'}) (\underline{L}(T) - \underline{L}(0)) + \gamma (\dot{\underline{L}}(T) - \dot{\underline{L}}(0)) \qquad (68)$$

$$= - \underline{\Delta E},$$

so that finally,

$$\underline{R}_{10}' = \underline{R}_{10} + (\alpha + \frac{\mu'}{\mu + \mu'}) (\underline{L}(T) - \underline{L}(0)) + \gamma (\dot{\underline{L}}(T) - \dot{\underline{L}}(0)). \qquad (69)$$

As before, $\underline{P}$, which may now be regarded as $\dot{\underline{R}}_{10}$ in the fixed system, is unmodified. The two-fixed center problem, with $\underline{R}_{10}'$ and $\underline{R}_{10}$ as initial conditions with the moon fixed at $\underline{L}(T)$ relative to the earth, should produce, at time T, a good approximation to the restricted problem, with initial condition $\underline{R}_{10}$ and $\underline{R}_{10}$ and the moon initially at $\underline{L}(0)$, provided T is small enough so that the second and higher order time derivatives of J_2 produce a negligible effect.

PRELIMINARY NUMERICAL RESULTS

The parameters α and γ have been determined for a lunar orbit with the following initial conditions:

$$x_{10} = -37163.688 \text{ km}$$
$$y_{10} = -56452.867 \text{ km}$$
$$z_{10} = -30844.317 \text{ km}$$
$$\dot{x}_{10} = -0.65536162 \text{ km/sec}$$
$$\dot{y}_{10} = -2.7369109 \text{ km/sec}$$
$$\dot{z}_{10} = -1.0459904 \text{ km/sec}$$

The distance of the vehicle from the earth is about 11.6 earth radii, and it has a speed of about 3 km/sec. For these conditions, the values of α and γ are the following:

$$\alpha = -6.2611792 \times 10^{-4}$$
$$\gamma = 0.28110731 \text{ hr}$$

The two-fixed-center calculation with the initial conditions modified for evaluation of the position and velocity of the vehicle at 23, 33, and 53 hours was compared with the base orbit at 23, 33, and 53 hours respectively. The deviations in position of the two-fixed-center calculation from the base case are shown in the table below. Included in the same table are the deviations of the corresponding Kepler problem from the base case.

Time	Dist. from Earth	Deviation	Two-Fixed -Center	Kepler
23 hr	35.3 ER	Δx	144 km	170 km
		Δy	132 km	200 km
		Δz	33 km	10 km
33 hr	42.1 ER	Δx	262 km	430 km
		Δy	155 km	250 km
		Δz	142 km	30 km
53 hr	52.7 ER	Δx	1300 km	1970 km
		Δy	1080 km	1100 km
		Δz	993 km	110 km

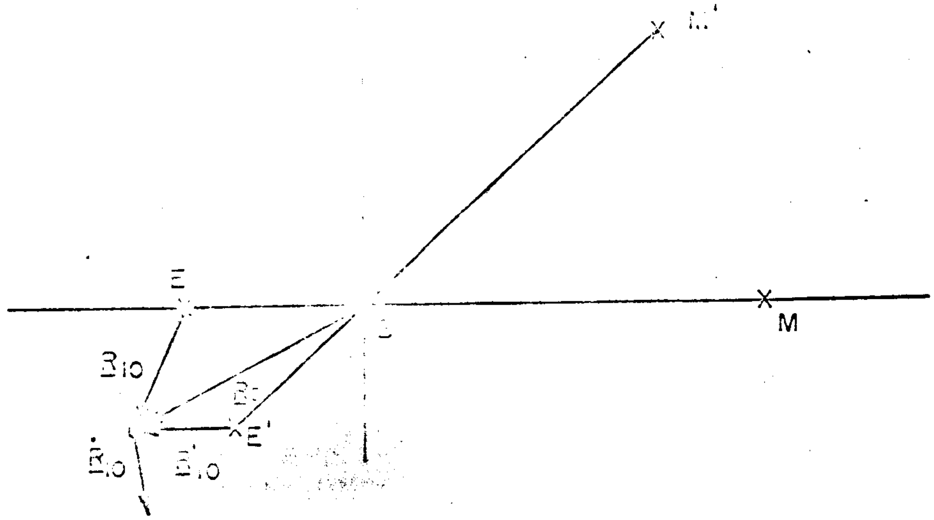

FIGURE 2

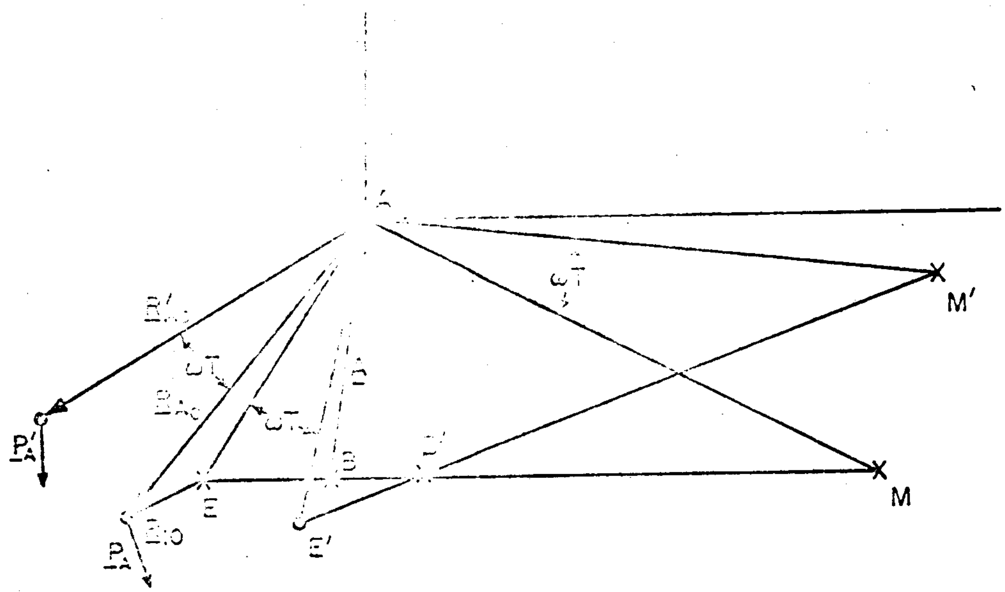

FIGURE 3

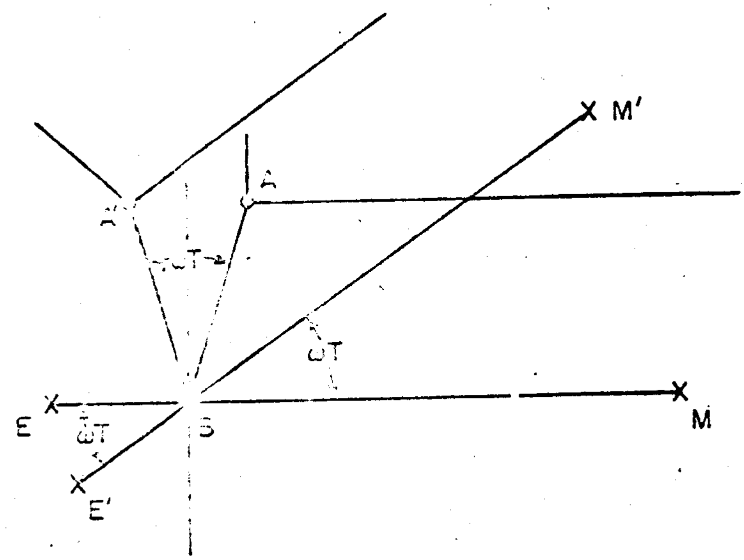

FIGURE 4

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Part I of

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APPLICATION OF THE TWO FIXED CENTER PROBLEM TO LUNAR TRAJECTORIES

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SUMMARY

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Six methods for the approximation of lunar trajectories by the two fixed center problem are developed. Four of these methods arise from a formulation of the restricted problem in a rotating coordinate system. The origin of the rotating system, to be regarded as the center of rotation is to be so selected as to improve the degree of approximation. The other two are developed from a formulation in an inertial system with fictitious fixed positions of the earth and moon selected so as to improve the approximation.

The results of a numerical comparison of the six methods with a typical lunar trajectory and the Kepler predictions are presented. These results are discussed and some suggestions are made for further development of the theory.

Author

LIST OF SYMBOLS

R	Position vector of vehicle relative to the earth-moon barycenter
R_1	Position vector of the vehicle relative to the earth
r_1	Distance of vehicle from earth
R_2	Position vector of the vehicle relative to the moon
r_2	Distance of vehicle from moon
μ	Gravitational constant times mass of the earth
μ'	Gravitational constant times mass of the moon
L	Position vector of moon relative to the earth
ℓ	Distance of moon from earth
$\dot{L}$	Velocity vector of the moon relative to the earth
Ω	Angular velocity vector of the moon relative to the earth
ω	Magnitude of Ω
$A=\alpha L+\beta\Omega+\gamma\dot{L}$	Origin, relative to the barycenter, of the rotating coordinate system
α, β, γ	Constants relating A to L, Ω and $\dot{L}$
$A_1=A-\beta\Omega$	Projection of A on the plane of the moon's motion
R_A	Position vector of the vehicle relative to A
H_A	Hamiltonian for restricted problem in a coordinate system rotating with angular velocity Ω about A
$\bar{R}_A$	Position vector of vehicle relative to A in the rotating system
$\bar{P}_A$	Momentum vector conjugate to $\bar{R}_A$

H_E	Hamiltonian for the Euler, or two fixed center problem
H_1	Perturbation Hamiltonian
$M(t)$	Rotation matrix through an angle $-\omega t$ about Ω
δ	A parameter introduced to improve minimization of the effect of the non-integrable terms in the perturbation equations
H	Hamiltonian in the inertial system
P	Momentum conjugate to R in the inertial system

SUBSCRIPTS

E	Refers to Euler problem
R	Refers to restricted problem
o	Refers to initial value
F	Refers to final value

NOTE: In general, capital letters represent vectors and the corresponding small letters their magnitudes. Bars over vectors denote vectors in a rotating coordinate system.

INTRODUCTION

In this report two general methods of obtaining approximations to the three dimensional restricted problem in terms of the two fixed center problem will be discussed in detail. The first method is based on a formulation of the restricted problem in a rotating coordinate system and the second on a formulation in an inertial system. In both methods perturbation equations are obtained for the initial conditions of the two fixed center problem regarded as osculating time varying parameters for the restricted problem. Both of these methods represent generalizations of a method developed by Arenstorf for treating the two dimensional restricted problem in a coordinate system rotating about the barycenter of the earth and moon.

The present formulation in the rotating system involves the selection of four scalar parameters in such a way as to reduce the effects of the non-integrable terms in the perturbation equations. Three of these parameters define the origin, to be regarded as the center of rotation, of the rotating system. The fourth allows part of one of the integrable terms to be used to reduce the effect of some of the non-integrable terms. A method for the determination of these four parameters is presented, and a set of osculating initial conditions is obtained by an approximate integration of the perturbation equations. In addition to this set three other sets are obtained by variations in the values of these parameters. In all of the methods developed the center of rotation is close to the center of the earth if the portion of the restricted orbit to be approximated has a close approach to the earth and no close approach to the moon. The center of rotation is close to the moon if the portion of the restricted orbit has a close approach to the moon and not to the earth. For midcourse portions, the center of rotation is somewhere between the earth and the moon. No attempt has so far been made to extend the theory to the approximation of portions containing close approaches to both the earth and the moon.

The formulation in the inertial system makes use of fictitious fixed positions for the earth and moon, so selected as to reduce the effect of the non-integrable terms in the perturbation equations. Two sets of formulas result which differ in the approximations used in the integration of the perturbation equations.

Altogether, then, six schemes are developed for approximating the restricted problem by the two fixed center problem. These schemes have been tested numerically for various portions of a typical lunar trajectory obtained by numerical integration. Some results of this numerical comparison are presented, following the analytical treatment.

The comparison shows clearly that the formulations in the rotating system are superior and the reasons for this are discussed in the last section.

THEORY FOR THE ROTATING SYSTEM

Derivation of the Perturbation Equations

The equations of motion of the restricted problem in an inertial system with origin at the barycenter are

$$\ddot{R} = -\mu \frac{R_1}{r_1^3} - \mu' \frac{R_2}{r_2^3} \tag{1}$$

Consider a point A defined by

$$A = \alpha L + \beta \Omega + \gamma \dot{L} \tag{2}$$

where L and $\dot{L}$ are position and velocity vectors of the moon relative to the earth, and hence are known functions of time satisfying the relation

$$\ddot{L} = -(\mu+\mu') \frac{L}{\ell^3} = \Omega \times (\Omega \times L) \qquad A_1 = \alpha L + \gamma \dot{L} \tag{3}$$

$$\dot{L} = \Omega \times L \qquad \ddot{A} = \Omega \times (\Omega \times A) = -\omega^2 A_1$$

$$\Omega = \frac{L \times \dot{L}}{\ell^2}$$

The point A thus rotates about the barycenter with the earth and the moon. The equations of motion for the restricted problem in an accelerated coordinate system with origin at A, but with axes always parallel to those of the inertial system, are

$$\ddot{R}_A = -\mu \frac{R_1}{r_1^3} - \mu' \frac{R_2}{r_2^3} - \ddot{A} \tag{4}$$

since

$$R = A + R_A, \quad \dot{R} = \dot{A} + \dot{R}_A, \quad \ddot{R} = \ddot{A} + \ddot{R}_A \tag{5}$$

and finally in a coordinate system rotating about A with angular velocity Ω the equation of motion become

$$\ddot{\bar{R}}_A = -\mu \frac{\bar{R}_1}{r_1^3} - \mu' \frac{\bar{R}_2}{r_2^3} - \ddot{\bar{A}} - \Omega \times (\Omega \times \bar{R}_A) - 2(\Omega \times \dot{\bar{R}}_A) \tag{6}$$

where bars denote vectors in the rotating system. We assume that at time $t = 0$, the axes of the rotating system are parallel to those in the inertial system, so that the constant vectors $\bar{A}$, $\dot{\bar{A}}$ and $\ddot{\bar{A}}$ satisfy the relations

$$\bar{A} = A\ (0) \qquad \dot{\bar{A}} = \dot{A}(0) \qquad \ddot{\bar{A}} = -\omega^2 \bar{A}_1 = -\omega^2 A_1(0) \tag{7}$$

It is readily verified that the Hamiltonian

$$H_A = \frac{1}{2} \bar{P}_A^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} - \omega^2 \bar{R}_A \cdot A_1 - \Omega \cdot \bar{R}_A \times \bar{P}_A \tag{8}$$

is a Hamiltonian for the problem represented by Eq. (6) with

$$\dot{\bar{R}}_A = \mathrm{grad}_{P_A} H_A = \bar{P}_A - \Omega \times \bar{R}_A \tag{9}$$

and

$$\dot{\bar{P}}_A = \ddot{\bar{R}}_A + \Omega \times \dot{\bar{R}}_A = -\mathrm{grad}_{\bar{R}_A} H_A$$

$$= -\mu \frac{R_1}{r_1^3} - \mu' \frac{R_2}{r_2^3} + \omega^2 \bar{A}_1 - \Omega \times \bar{P}_A$$

$$= -\mu \frac{R_1}{r_1^3} - \mu' \frac{R_2}{r_2^3} + \omega^2 \bar{A}_1 - \Omega \times \dot{\bar{R}}_A - \Omega \times (\Omega \times \bar{R}_A) \tag{10}$$

which reduces immediately to Eq. (6). A word on the relation between position in the rotating system $\bar{R}_A$ and its conjugate momentum $\bar{P}_A$ and the position R_A and velocity $\dot{R}_A$ in the non-rotating system is necessary for the interpretation of results to be obtained later. Since the rotating and non-rotating systems are assumed coincident at $t = 0$

$$\begin{aligned} \bar{R}_{A0} &= R_{A0} \\ \bar{P}_{A0} &= \dot{\bar{R}}_{A0} + \Omega \times \bar{R}_{A0} = \dot{R}_{A0} \end{aligned} \tag{11}$$

are vector equations which are valid component by component, and since $\dot{\bar{R}}_{A0}$ is the velocity relative to A in the rotating system while $\Omega \times \bar{R}_{A0}$ is the velocity due to the rotation of the system it is seen that the initial value of the momentum conjugate to $\bar{R}_A$ is just the velocity in the non-rotating system. The same statements hold for time t also, except that to get component agreement a rotation through ωt is necessary. That is, at time t

$$R_A = M^{-1}(t)\,\bar{R}_A$$
$$\dot{R}_A = M^{-1}(t)\,\bar{P}_A = M^{-1}(t)\,(\dot{\bar{R}}_A + \Omega \times \bar{R}_A) \tag{12}$$

where $M^{-1}(t)$ may be regarded either as a rotation of the axes of the rotating system through an angle $-\omega t$ or as a rotation of $\bar{R}_A$ and $\bar{P}_A$ relative to the rotating axes through and angle ωt, both rotations about the vector Ω which is the same in both systems.

The Hamiltonian H_A may be written as the sum of

$$H_E = \frac{1}{2}\bar{P}_A^{\,2} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \tag{13}$$

the Hamiltonian for the Euler problem with Hamilton equation

$$\dot{\bar{R}}_{AE} = \text{grad}_{\bar{P}_{AE}}\, H_E = \bar{P}_{AE}$$
$$\dot{\bar{P}}_{AE} = \ddot{\bar{R}}_{AE} = -\mu \frac{R_1}{r_1^{\,3}} - \mu' \frac{R_2}{r_2^{\,3}} \tag{14}$$

and a perturbation

$$H_1 = -\omega^2\, \bar{R}_A \cdot \bar{A}_1 - \Omega \cdot \bar{R}_A \times \bar{P}_A \tag{15}$$

where the subscript E in Eqs. (14) refers to the functional forms for $\bar{R}_A$ and $\bar{P}_A$ obtained by solving Eqs. (14).

A solution of the restricted problem with Hamiltonian given by Eq. (8) and Hamiltonian Eqs. (9) and (10) is now sought in the functional form of the solution of the Euler problem with time varying initial conditions. That is, one seeks the solution of the restricted problem, denoted by a subscript R, in the form

$$\bar{R}_{AR}(\bar{R}_{AR0}, \bar{P}_{AR0}, t) = \bar{R}_{AE}\left(\bar{R}_{AE0}(t), \bar{P}_{AE0}(t), t\right)$$
$$\bar{P}_{AR}(\bar{R}_{AR0}, \bar{P}_{AR0}, t) = \bar{P}_{AE}\left(\bar{R}_{AE0}(t), \bar{P}_{AE0}(t), t\right) \tag{16}$$

with initial conditions for the restricted and Euler problems satisfying the relations

$$\bar{R}_{AR}(\bar{R}_{AR0}, \bar{P}_{AR0}, 0) = \bar{R}_{AR0} = \bar{R}_{AE}\left(\bar{R}_{AE0}(0), \bar{P}_{AE0}(0), 0\right) = \bar{R}_{AE}(0)$$
$$\bar{P}_{AR}(\bar{R}_{AR0}, \bar{P}_{AR0}, 0) = \bar{P}_{AR0} = \bar{P}_{AE}\left(\bar{R}_{AE0}(0), \bar{P}_{AE0}(0), 0\right) = \bar{P}_{AE0}(0) \tag{17}$$

It has been shown by Arenstorf[1] that the functions $\bar{R}_{AE0}(t)$ and $\bar{P}_{AE0}(t)$ necessary for the validity of Eq. (16) satisfy the differential equations

$$\frac{d}{dt}\bar{R}_{AE0}(t) = \mathrm{grad}_{\bar{P}_{AE0}}\bar{H}_1$$
$$\frac{d}{dt}\bar{P}_{AE0}(t) = -\,\mathrm{grad}_{\bar{R}_{AE0}}\bar{H}_1 \tag{18}$$

where

$$\bar{H}_1 = \bar{H}_1\left(\bar{R}_{AE0}(t), \bar{P}_{AE0}(t), t\right) \tag{19}$$

is obtained by substitution of $\bar{R}_{AE}\left(\bar{R}_{AE0}(t), \bar{P}_{AE0}(t), t\right)$ and $\bar{P}_{AE}\left(\bar{R}_{AE0}(t), \bar{P}_{AE0}(t), t\right)$ for $\bar{R}_A$ and $\bar{P}_A$ in H_1 (given by Eq. (15). To actually carry out the substitution using the solution of the Euler problem (which is known in closed form) and then compute the gradients required in Eq. (18) would be very complex because of the extreme complexity of the closed form solution. Even could this be carried out the integration of the resulting highly nonlinear equations in $\bar{R}_{AE0}(t)$ and $\bar{P}_{AE0}(t)$ would be very difficult. Further, any approximation method for integration of perturbation equations for initial conditions must be developed with great care to avoid the introduction of troublesome secular terms, which increase in order with higher order approximations.

In view of this last fundamental difficulty, only a first approximation will be attempted. This approximation will lead to some integrable terms in the perturbation equations and the point A will be selected in such a way as to reduce the effect of the non-integrable terms, which will then be ignored. The resulting expressions for the time variation in the initial conditions and hence the solution of the restricted problem represented by Eq. (16) will thus have limited validity in time. The hardest part of the problem will be in obtaining an estimate for duration of validity. Although this might appear to restrict considerably the application of the theory, it should nevertheless be noted that from the solutions of a sequence of two fixed center problems, each valid for a certain time, the solution of the restricted problem may be constructed solely in terms of closed form calculations without the use of numerical integration. Such a procedure will be outlined later.

Explicit Form of the Perturbation Equations

To proceed with the approximation $\bar{H}_1$ is written in the form

$$\bar{H}_1 = - \Omega \cdot \bar{R}_{AE0} \times \bar{P}_{AE0} - \omega^2 \delta \, \bar{R}_{AE0} \cdot \bar{A}_1 - \omega^2 (1-\delta) \, \bar{R}_{AE} \cdot \bar{A}_1$$
$$- \int \left\{ \omega^2 \delta \, \bar{P}_{AE} \cdot \bar{A} - \Omega \cdot \bar{R}_{AE} \times \left(\mu \frac{\bar{R}_1}{r_1^3} - \mu' \frac{\bar{R}_2}{r_2^3} \right) \right\} dt \tag{20}$$

where the integral is obtained by time differentiation of $(-\omega^2 \delta \, \bar{R}_{AE} \cdot \bar{A}_1 - \Omega \cdot \bar{R}_{AE} \times \bar{P}_{AE})$ and use of the Hamilton Eqs. (14) for the Euler problem. The first two terms of $\bar{H}_1$ will be shown to lead to integrable terms in the perturbation equations (18) for the initial conditions. The factor δ permits part of the $R_A \cdot A_1$ term to appear with the integrable terms and part with the non-integrable terms. This second part helps to reduce the effect of the other non-integrable terms. The third term and the integral are not written explicitly in terms of initial conditions. It is these terms for which an effort at minimization will be made by proper selection of the factor δ and the point A. To see how this may be done one now takes the gradients of $\bar{H}_1$ with respect to $\bar{R}_{AE0}$ and $\bar{P}_{AE0}$ to obtain the perturbation equations. The differentiation of the triple product in the integral is facilitated by noting that

$$\bar{R}_1 = \bar{R}_A + \bar{A} + \frac{\mu'}{\mu+\mu'}\bar{L} \qquad \bar{R}_2 = \bar{R}_A + \bar{A} \frac{\mu}{\mu+\mu'}\bar{L} \tag{21}$$

so that

$$\begin{aligned} \bar{R}_A \times \bar{R}_1 &= \bar{R}_A \times \left(\bar{A} + \frac{\mu'}{\mu+\mu'}\bar{L}\right) \\ \bar{R}_A \times \bar{R}_2 &= \bar{R}_A \times \left(\bar{A} - \frac{\mu}{\mu+\mu'}\bar{L}\right) \end{aligned} \tag{22}$$

The perturbation equations for the time derivatives of $\bar{R}_{AE0}(t)$ and $\bar{P}_{AE0}(t)$ are readily verified to be

$$\begin{aligned} \frac{d}{dt} R_{AE0}(t) &= \text{grad}_{PAE0}\bar{H}_1 = -\Omega \times R_{AE0} - \omega^2(1-\delta)\Psi_{RP} A_1 \\ &\quad - \int\left[\omega^2 \delta \Psi_{PP} A_1 - \Psi_{RP}(\Omega \times M - \omega Q)\right] dt \end{aligned} \tag{23}$$

and

$$\begin{aligned} \frac{d}{dt} P_{AE0}(t) &= -\text{grad}_{RAE0}\bar{H}_1 = -\Omega \times P_{AE0} + \omega^2 \delta A_1 + \omega^2(1-\delta)\Psi_{RR} A_1 \\ &\quad + \int\left[\omega^2 \delta \Psi_{PR} A_1 - \Psi_{RR}(\Omega \times M - \omega Q)\right] dt \end{aligned} \tag{24}$$

where the M and Q are vectors given by

$$\begin{aligned} M &= \mu \frac{\bar{A} + \frac{\mu'}{\mu+\mu'}\bar{L}}{r_1^{\,3}} + \mu' \frac{\bar{A} - \frac{\mu}{\mu+\mu'}\bar{L}}{r_2^{\,3}} \\ Q &= \frac{3}{\omega}\left\{\mu\left[\Omega \cdot \bar{R}_{AE} \times \left(\bar{A} + \frac{\mu'}{\mu+\mu'}\bar{L}\right)\frac{\bar{R}_1}{r_1^{\,5}}\right]\right. \\ &\quad \left. + \mu'\left[\Omega \cdot \bar{R}_{AE} \times \left(\bar{A} - \frac{\mu}{\mu+\mu'}\bar{L}\right)\frac{\bar{R}_2}{r_2^{\,5}}\right]\right\} \end{aligned} \tag{25}$$

These vectors are so defined that they have the same dimension. The Ψ's are matrices given by

$$\Psi_{RR} = \left(\frac{\partial R_{AEj}}{\partial R_{AE0i}}\right) \qquad \Psi_{RP} = \left(\frac{\partial R_{AEj}}{\partial P_{AE0i}}\right)$$

$$\Psi_{PR} = \left(\frac{\partial P_{AEj}}{\partial R_{AE0i}}\right) \qquad \Psi_{PP} = \left(\frac{\partial P_{AEj}}{\partial P_{AE0i}}\right) \tag{26}$$

with the i^{th} row and j^{th} column containing the derivative of the j^{th} component of the time varying vector in the numerator with respect to the i^{th} component of the initial value vector in the denominator evaluated at $R_{AE0}(t)$ and $P_{AE0}(t)$. It may be noted that the transposes of these matrices constitute the transition matrix for the Euler problem with the transposes of the first two matrices forming the top three rows and the transposes of the last two matrices forming the bottom three rows.

Determination of the Origin A and the Parameter δ

The first term in the right hand side of Eq. (23) and the first two terms on the right side of Eq. (24) depend only on the initial values $R_{AE0}(t)$ and $P_{AE0}(t)$ and if these were the only terms present Eqs. (23) and (24) would be integrable. The remaining terms all involve components of the transition matrix for the Euler problem and no attempt will be made to include them in the integration. Instead methods will be sought for making them small, and this will be done by seeking an approximate minimization of the vectors on which the matrices operate. These vectors appear in both equations as follows:

$$\begin{aligned} N_1 &= \omega^2(1-\delta)\,\bar{A}_1 && \text{outside the integrals} \\ N_2 &= \omega^2\delta\,\bar{A}_1 && \text{inside the integrals} \end{aligned} \tag{27}$$

together with M and Q defined in Eqs. (25), which appear inside the integrals. It will be noted that all these vectors have the same dimension. The vectors M and Q are functions of time. Since however, they have, effectively, the cubes of r_1 and r_2 in the denominator, it is clear that they are large only for brief periods of time at approach to the earth or the moon closer than a few earth radii.

As a first trial at minimization, δ and A were sought such that the scalar σ

$$\sigma = N_1{}^2 + N_2{}^2 + M_o{}^2 + M_f{}^2 \tag{28}$$

is minimized, where M_o and M_f are computed from initial and anticipated final conditions, respectively. The omission of Q is heuristically justified by an argument of the following type. Suppose the initial position is close to the earth and the final position close to the moon. Initially the r_2 terms are small, so that to minimize the r_1 terms $\left(\bar{A} + \frac{\mu'}{\mu + \mu'} \bar{L}\right)$ must nearly vanish in order to keep M_o small. It will then follow that Q_o is also small. Evidently, of course, such a procedure will mean that both M_f and Q_f will become more or less large depending on the final value of r_2. In effect, this will place a limitation on the duration of validity of the two fixed center approximation.

The minimization of Eq. (28) will now be carried out. Since M and, for that matter Q also, are independent of δ, partial derivatives of σ with respect to δ involve only the N_1 and N_2 terms:

$$\frac{\partial \sigma}{\partial \delta} = N_1 \cdot \frac{\partial N_1}{\partial \delta} + N_2 \cdot \frac{\partial N_2}{\partial \delta} = \omega^4 \bar{A}_1^2 \Big(2\delta - 2(1-\delta) \Big) \tag{29}$$

which vanishes for $\delta = \frac{1}{2}$. It now remains to minimize

$$\sigma_1 = \frac{1}{2} \omega^4 \bar{A}_1^2 + M_o{}^2 + M_f{}^2 \tag{30}$$

with respect to α, β and γ. That is the equations

$$\frac{\partial \sigma_1}{\partial x} = \omega^4 \bar{A}_1 \cdot \frac{\partial \bar{A}_1}{\partial x} + M_o \cdot \frac{\partial M_o}{\partial x} + 2M_f \cdot \frac{\partial M_f}{\partial x} = 0 \tag{31}$$

where x denotes α, β and γ must be solved for α, β and γ. Recalling that

$$\bar{A} = \alpha \bar{L} + \beta \bar{\Omega} + \gamma \bar{\bar{L}} \quad , \qquad \bar{A}_1 = \alpha \bar{L} + \gamma \bar{\bar{L}} \tag{32}$$

one obtains the following:

$$\frac{\partial \bar{A}_1}{\partial \alpha} = \frac{\partial \bar{A}}{\partial \alpha} = \bar{L}, \quad \frac{\partial \bar{A}_1}{\partial \beta} = 0, \quad \frac{\partial \bar{A}}{\partial \beta} = \bar{\Omega}, \quad \frac{\partial A_1}{\partial \gamma} = \frac{\partial \bar{A}}{\partial \gamma} = \dot{\bar{L}} \tag{33}$$

and from the first of Eqs. (25) evaluated at initial and final positions, respectively:

$$\frac{\partial M}{\partial \alpha} = \bar{L}\left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right), \; \frac{\partial M}{\partial \beta} = \bar{\Omega}\left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right), \; \frac{\partial M}{\partial \gamma} = \dot{\bar{L}}\left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right) \tag{34}$$

so that

$$\bar{A}_\ell \cdot \frac{\partial \bar{A}_1}{\partial \alpha} = \alpha \ell^2, \qquad \bar{A}_\ell \cdot \frac{\partial \bar{A}_1}{\partial \beta} = 0, \qquad \bar{A}_\ell \cdot \frac{\partial \bar{A}_1}{\partial \gamma} = \gamma \dot{\ell}^2 \tag{35}$$

$$M \cdot \frac{\partial M}{\partial \alpha} = \alpha \left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right)^2 \ell^2 + \frac{\mu\mu'}{\mu+\mu'} \ell^2 \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)\left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right) \tag{36}$$

$$M \cdot \frac{\partial M}{\partial \beta} = \omega^2 \left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right)^2 \beta, \; M \cdot \frac{\partial M}{\partial \gamma} = \dot{\ell}^2 \left(\frac{\mu}{r_1^3} + \frac{\mu'}{r_2^3}\right)^2 \gamma \tag{37}$$

Substitution in Eq. (31) for $x = \beta$ and γ lead to

$$\beta = \gamma = 0 \tag{38}$$

while for $x = \alpha$, one obtains

$$\begin{aligned} &\omega^4 \alpha \ell^2 + 2\alpha \ell^2 \left[\left(\frac{\mu}{r_{10}^3} + \frac{\mu'}{r_{20}^3}\right)^2 + \left(\frac{\mu}{r_{1f}^3} + \frac{\mu'}{r_{2f}^3}\right)^2 \right] \\ &\quad + 2\mu \mu' \ell^2 \left[\left(\frac{1}{r_{10}^3} - \frac{1}{r_{20}^3}\right)\left(\frac{\mu}{r_{10}^3} + \frac{\mu'}{r_{20}^3}\right) \right. \\ &\quad \left. + \left(\frac{1}{r_{1f}^3} - \frac{1}{r_{2f}^3}\right)\left(\frac{\mu}{r_{1f}^3} + \frac{\mu'}{r_{2f}^3}\right) \right] = 0 \end{aligned} \tag{39}$$

or

$$\alpha = -\frac{\mu\mu'}{\mu+\mu'} \frac{\sum_{i=0,f} \left(\frac{1}{r_{1i}^3} - \frac{1}{r_{2i}^3}\right)\left(\frac{\mu}{r_{1i}^3} + \frac{\mu'}{r_{2i}^3}\right)}{\left[\sum_{i=0,f} \left(\frac{\mu}{r_{1i}^3} + \frac{\mu'}{r_{2i}^3}\right)^2\right] + \frac{1}{2}\omega^4} \tag{40}$$

Some comments on the value of α may be made. If a close approach only to the earth is made, that is if either r_{10} or r_{1f} is close to unity while r_{20} and r_{2f} are both large it is readily seen that

$$\alpha \sim -\frac{\mu'}{\mu + \mu'},$$

which corresponds to placing the origin at the earth, while if a close approach only to the moon is made

$$\alpha \sim \frac{\mu}{\mu + \mu'}$$

which corresponds to placing the origin at the moon. If a midcourse portion of the trajectory is to be approximated so that none of the r's is near unity α will be somewhere between these extreme values -- that is the origin will lie on the line of centers between the earth and the moon. The origin is at the barycenter for $\alpha = 0$.

Integration of the Perturbation Equations

Once the point A has been determined the non-integrable terms in the perturbation equations (23) and (24) will be ignored and the equations to be integrated are

$$\frac{d}{dt} \bar{R}_{AE0}(t) = - \bar{\Omega} \times \bar{R}_{AE0}(t) \tag{41}$$

$$\frac{d}{dt} \bar{P}_{AE0}(t) = - \bar{\Omega} \times \bar{P}_{AE0}(t) + \omega^2 \delta \bar{A} \tag{42}$$

$$= - \bar{\Omega} \times \left(\bar{P}_{AE0}(t) + \delta \bar{\Omega} \times \bar{A} \right)$$

where use has been made of the relation

$$\bar{\Omega} \times (\bar{\Omega} \times \bar{A}) = - \omega^2 \bar{A} \tag{43}$$

The integrals of these equations are, since $\delta \bar{\Omega} \times \bar{A}$ is a constant,

$$\bar{R}_{AE0}(t) = M(t)\,\bar{R}_{AE0}(0) \tag{44}$$

$$\bar{P}_{AE0}(t) = M(t)\left[\bar{P}_{AE0}(0) + \delta\,\bar{\Omega} \times \bar{A}\right] - \delta\,\bar{\Omega} \times \bar{A} \tag{45}$$

where the matrix M(t) is a rotation matrix through an angle $-\omega t$ about the $\bar{\Omega}$ direction.

Referring back, now, to Eq. (16), it is seen that, in the rotating system, a solution to the restricted problem valid from the initial time zero to some time t, determined by how long the nonintegrable terms remain negligible, is obtained by substitution of the expressions (44) and (45) for $\bar{R}_{AE0}(t)$ and $\bar{P}_{AE0}(t)$ in terms of the two fixed center problem. This means that in order to construct the solution of the restricted problem in terms of that of the two fixed center problem, it is necessary, for each time t of interest, to compute initial conditions from Eqs. (44) and (45), and then obtain the solution,evaluated at the time t, of a two fixed center problem with these initial conditions. Thus if n points on the restricted orbit are desired, n different two fixed center problems must be evaluated.

One other point should be mentioned. The initial value $\bar{P}_{AE0}(0)$ is to be thought of as given by $\bar{P}_{AR0}$, which in turn is determined by the first Hamilton equation (9) for the restricted problem evaluated at time t=0:

$$\bar{P}_{AR0} = \dot{\bar{R}}_{A0} + \Omega \times \bar{R}_{A0} = \dot{R}_{A0} \tag{46}$$

where $\dot{R}_{A0}$ is the initial velocity in the non-rotating system, since the assumption has been made that the rotating and non-rotating systems have parallel axes at the initial time. Once $\bar{P}_{AE0}(0)$ has been been determined $\bar{P}_{AE0}(t)$ is given by Eq. (45) and is to be interpreted as an initial velocity relative to $\bar{A}$ in the rotating system for the two fixed center problem, by virtue of the first of the Hamilton equations (14) for this problem. Since in the rotating system the earth and moon are fixed the initial velocity $\bar{P}_{AE0}(t)$ is the same relative to any point in this system. The two fixed center solution obtained from this initial velocity $\bar{P}_{AE0}(t)$ and the initial position $\bar{R}_{AE0}(t)$ lead to position $\bar{R}_{AE}(t)$ and velocity $\bar{P}_{AE}(t)$ for the two fixed center problem, which are to be interpreted as position $\bar{R}_{AR}(t)$ and momentum $\bar{P}_{AR}(t)$ for the restricted problem in the rotating system.

THEORY FOR THE INERTIAL SYSTEM

Derivation and Integration of the Perturbation Equations

A direct approach to an approximation of the solution of the restricted problem by the two fixed center problem in an inertial coordinate system can be developed as follows. Recalling the equations of motion for the restricted problem in the inertial system with origin at the barycenter.

$$\ddot{R} = -\mu \frac{R_1}{r_1^3} - \mu' \frac{R_2}{r_2^3} \tag{1}$$

it is easily shown that the Hamiltonian is

$$H = \frac{1}{2} \dot{P}^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \tag{47}$$

This Hamiltonian has an explicit time dependence since r_1 and r_2 are distances of the vehicle from the earth and moon which are assumed moving in known orbits about the barycenter. The momentum P conjugate to position R relative to the varycenter is just $\dot{R}$, the velocity relative to the barycenter. The first Hamilton equation expresses this fact, and the second, together with the first, yields the equations of motion (1).

In this formulation two fixed points are selected for a fixed earth and a fixed moon. The selection of these points is to be made so as to minimize the non-integrable portion of the perturbation equations. Thus, denoting positions relative to these fixed points by stars, the equations of motion are

$$\ddot{R} = -\mu \frac{R_1^*}{r_1^{*3}} - \mu' \frac{R_2^*}{r_2^{*3}} + \mu\left(\frac{R_1^*}{r_1^{*3}} - \frac{R_1}{r_1^3}\right) + \mu'\left(\frac{R_2^*}{r_2^{*3}} - \frac{R_2}{r_2^3}\right) \tag{48}$$

and the Hamiltonian is

$$H = \frac{1}{2} P^2 - \frac{\mu}{r^*_1} - \frac{\mu'}{r^*_2} + \mu\left(\frac{1}{r^*_1} - \frac{1}{r_1}\right) + \mu'\left(\frac{1}{r^*_2} - \frac{1}{r_2}\right) \tag{49}$$

The Hamiltonian can be expressed as the sum of two terms. The first is the Hamiltonian for the two fixed center problem

$$H_E = \frac{1}{2} P^2 - \frac{\mu}{r^*_1} - \frac{\mu'}{r^*_2} \tag{50}$$

and the second

$$H_1 = \mu\left(\frac{1}{r^*_1} - \frac{1}{r_1}\right) + \mu'\left(\frac{1}{r^*_2} - \frac{1}{r_2}\right) \tag{51}$$

is the perturbation Hamiltonian which may be written in the form

$$H_1 = \mu\left(\frac{1}{r^*_{10}} - \frac{1}{r_{10}}\right) + \mu'\left(\frac{1}{r^*_{20}} - \frac{1}{r_{20}}\right) \tag{52}$$

$$-\int\left\{\mu\left[\frac{R^*_1 \cdot \dot{R}^*_1}{{r^*_1}^3} - \frac{R_1 \cdot \dot{R}_1}{r_1^3}\right] + \mu'\left[\frac{R^*_2 \cdot \dot{R}^*_2}{{r^*_2}^3} - \frac{R_2 \cdot \dot{R}_2}{r_2^3}\right]\right\} dt$$

Perturbation equations for the initial conditions may now be written as

$$\frac{d}{dt} R_o(t) = \text{grad}_{Po} H_1 = 0 - \text{grad}_{Po} \int \{\cdots\}\, dt \tag{53}$$

$$\frac{d}{dt} P_o(t) = -\text{grad}_{Ro} H_1 = \mu\left(\frac{R_{10}^*}{{r_{10}^*}^3} - \frac{R_{10}}{r_{10}^3}\right) + \mu'\left(\frac{R_{20}^*}{{r_{20}^*}^3} - \frac{R_{20}}{r_{20}^3}\right) + \text{grad}_{Ro} \int \{\cdots\}\, dt$$

If the terms involving the integrals are ignored in the perturbation equations, one obtains

$$R_o(t) = R_o(0) \tag{54}$$

$$P_o(t) = P_o(0) + (t - t_o)\left[\mu\left(\frac{R_{10}^*}{{r_{10}^*}^3} - \frac{R_{10}}{{r_{10}^*}^3}\right) + \mu'\left(\frac{R_{20}^*}{{r_{20}^*}^3} - \frac{R_{20}}{r_{20}^3}\right)\right]$$

since the first of these equations implies also

$$R_{io}^*(t) = R_{io}^*(o) \qquad R_{io}(t) = R_{io}(o) \qquad i = 1, 2 \tag{55}$$

<u>Selection of Fixed Positions for Earth and Moon</u>

It is not easy to see how the fixed positions for the earth and moon should be selected so as to minimize the contribution of the integrals to the perturbation equations (53). Examination of the equations of motion (48), however, suggests that two cases should be considered as follows:

1) Motion from earth towards moon; fix earth in its initial and moon in its final position.

2) Motion from moon towards earth; fix earth in its final and moon in its initial position.

The initial conditions for the two fixed center problem will then be determined by the condition that initial position relative to the barycenter is unmodified and initial velocity relative to the barycenter be determined from Eq. (54), with momentum identified with velocity. The solution $R_E(R_o, P_o(t), t)$ and $P_E(R_o, P_o(t), t)$ for the Euler problem will then be related to that for the restricted problem by

$$R_R(R_o, P_o, t) = R_E(R_o, P_o(t), t) \tag{56}$$

$$P_R(R_o, P_o, t) = P_E(R_o, P_o(t), t)$$

where R_R and P_R are to be interpreted as position and velocity relative to the barycenter at time t.

RESULTS OF NUMERICAL COMPARISONS

Two methods of approximating the restricted problem by the two fixed center problem have been obtained in the preceding two sections. In addition to these methods, three others based on the formulation in the rotating system have been considered. These last three methods are defined as follows:

A. The center of rotation is taken at the center of the moon if the portion of a lunar trajectory to be approximated lies in "moon reference"; that is, if all points on this portion are within about 9 earth radii of the moon. For portions of the trajectory outside moon reference the center of rotation is taken at the earth. The method has not been applied to portions of a lunar trajectory crossing the moon's sphere of influence. Thus the values of α used for method A:

$$\alpha = -\frac{\mu'}{\mu+\mu'} \qquad \text{earth reference}$$

$$\alpha = \frac{\mu}{\mu+\mu'} \qquad \text{moon reference}$$

are the two extreme values noted in the discussion following Eq. (40) for α.

In addition, the parameter δ is taken to be zero.

B. This method uses the value of α determined by Eq. (40). The value of δ is taken to be one.

C. This method also uses the value of α given by Eq. (40), and δ is set equal to zero.

The two methods already derived are identified by

D. The method in the rotating system.

E. The method in the inertial system.

F. Finally, a sixth method was tried in which the effect of the perturbation Hamiltonian in the inertial formulation was neglected. That is the initial conditions for the two fixed center problem are to be just the initial position and velocity relative to the barycenter.

The comparison of the effectiveness of these methods was carried out as follows. First a typical lunar trajectory was integrated with the effects of moving earth and moon included, but with all perturbations due to sun, other planets, oblateness etc. eliminated from the program. The integration was carried out by the Republic Interplanetary Program using the Encke method. In this program the earth is used as origin in earth reference and the moon is the origin in moon reference. Various points on this typical lunar trajectory were taken as initial points and the two fixed center approximation was computed at various specified later times. This necessitated the transformation of the initial conditions associated with the various methods (relative to the origin A for the rotating formulations and relative to the barycenter for the inertial formulations) into equivalent initial conditions relative to the earth or moon for portions of the trajectory in earth and moon reference respectively.

The base lunar trajectory started at time t=0 from about 6590 Km from the center of the earth, reached a perisel distance of about 4350 Km at 71 hr. and reached a perigee distance of 8174 Km at 153.9 hr. The entry and exit from moon reference occurred at about 58.7 hr. and 84.1 hr. respectively.

Tables I, II, III and IV contain some typical results from the numerical calculations. Tables I and IV are for the earth-reference portions of the trajectory on the first and last legs, respectively. Tables II and III are for moon reference portions approaching and receding from the moon, respectively. The left hand column contains the initial and final times for the portion of the trajectory to be approximated. The deviations Δx, Δy and Δz in kilometers for the various methods are entered in columns headed by the corresponding letter. These deviations represent the difference in the rectangular coordinates relative to the reference body, the values predicted by the various methods being subtracted from the values given by the base case. The column headed K, which appears in Tables I and IV, give the deviations for the Kepler problem. The last column gives the value of α determined from Eq. (40) for use in methods B, C and D.

In Table V the x, y and z coordinates of the vehicle relative to the reference body are given for the various times which appear in Tables I, II, III and IV. Also given are the distances of the vehicle from the reference body in earth radii. The distance of the earth from the moon is a little less than 60 E.R.

Some general conclusions on the relative merits of these methods may be drawn. First it may be noted that methods A and C are practically the same except for midcourse portions of the trajectory. The reason for this is that except for such portions the value of α is such that the origin is nearly at the earth for earth reference and nearly at the moon for moon reference.

To summarize the results, then, for the methods described in this report

A and C are best for long range on the return leg.

B and C have a slight superiority for midcourse.

D is best in moon reference, on the first leg and for short range on the return leg.

E and F are inferior almost everywhere.

The Kepler problem is superior to all of these methods for short to medium range in the neighborhood of the earth and moon. It fails, however, for long range and midcourse portions of the trajectory.

CONCLUSIONS

The results of the numerical comparison made in the previous section show that the formulation in a rotating system is best suited to the approximation of the restricted problem by the two fixed center problem. This is not really very surprising because in a rotating system the earth and moon are automatically fixed. This is achieved by introducing terms corresponding to the centrifugal and Coriolis accelerations, which are interpreted as perturbations on the two fixed center problem. In the inertial system, on the other hand, fixed positions for the earth and moon had to be selected more or less arbitrarily. As a consequence the perturbations from the two fixed center problem so selected depends on this selection. Thus approximations have been introduced before the problem of approximating the effect of the perturbations can even be considered. It would therefore seem that a rotating system, in which only the problem of how to treat the perturbations appears, should be the proper choice.

From the numerical results shown in the last section, it is evident that the problem of treating the perturbations is far from an easy one. None of the numerical results obtained can be regarded as satisfactory, or, in fact, as fulfilling the expectations that one might have for the theory. Nevertheless, there are a number of reasons for expecting that further development of the theory should lead to useful and interesting results.

If, for example, one considers the determination of the origin for the rotating system, it is obvious that the method used is fairly crude. The sum of squares of certain vectors appearing in the perturbation equations is minimized. Evidently, if the sum were a weighted sum, different origins would be obtained depending on the weighting factors used. It should, however, be remarked that the present determination yields plausible results, e.g., in the case of motion of an earth or moon satellite, one would certainly expect the rotation of initial conditions implied by Eqs. (44) and (45) to be about the center of the primary attracting body, or at least about a point very close to its center. A large rotation about a point very far removed from the center would obviously drastically distort what should be a stable orbit. Thus, the property that the origin is closer to the earth or moon according as the portion of the restricted problem orbit under consideration is closer to the earth or moon is a reasonable one and shows that the theory is at least qualitatively correct in this respect. For midcourse portions of the trajectory, one cannot use the satellite argument to suggest the proper choice of the origin, though it might be conjectured that the origin should vary continuously with the portion of the trajectory to be approximated.

It is possible to make a few remarks on the parameter δ. Reference to the perturbation Eqs. (23) and (24) shows that if $\delta = 1$ the non-integrable terms are all integrals from initial to final time, which therefore have zero initial value. It would thus appear that for short range predictions, results for $\delta = 1$, that is for method B, would be superior to the others. This result has been observed for some midcourse runs.

It may have been noticed that the perturbation term $\Omega \cdot R_A \times P_A$ in the perturbation Hamiltonian H_1 (see Eqs. (15) and (20) could be treated in the same way as the $R_A \cdot A_1$ term. That is, a factor ϵ could be introduced in the same way as δ. This would change the rotation in the initial conditions, resulting from integration of the perturbation equations, from an angle ωt to an angle $\epsilon \omega t$. To actually introduce the ϵ and obtain a value for it in the same way as for δ would not be easy because the terms in $(1-\epsilon)$ which would appear both inside and outside the integrals would be far more complex and difficult to treat than the corresponding terms in $(1-\delta)$.

To summarize, then, the various methods so far developed for the rotating system depend on the selection of four parameters α, β, γ (determining the center of rotation A) and δ. At this stage it appears that some sort of a parameter study using variations from the values of the parameters so far used, and including also, perhaps, variations in the parameter ϵ defined in the last paragraph, might well lead to some useful approximation formulae. There are many ways in which such a study might be carried out, for example, by using weighting factors with the vectors to be minimized, by a systematic variation of the parameters, or by the development of some sort of interation procedure. From the above discussion, it would appear that β and γ should be close to zero, that ϵ should be close to one, and that α should vary approximately according to Eq. (40). Only for the parameter δ is it difficult to estimate a value except for relatively short range predictions for which one would expect δ to be close to one.

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Table I. Earth Reference From Earth to Moon

α = -.012116806 Corresponds to Center of Rotation at the Earth

		A	B	C	D	E	F	K	α
0-1	Δx	.195	-.170	.177	.008			-.001	-.012116746
	Δy	.1169	-.0277	.1189	.0449			-.0074	
	Δz	.0426	-.0610	.0506	-.0045			.0002	
0-10	Δx	19	-52	19	-16	1033	1025	-6	-.012116746
	Δy	20	-28	20	-4	310	308	-4.8	
	Δz	8.4	-32	8.3	-1.692	471	467	-.65	
0-30	Δx		-523	125	-198	3431	3181	-174	-.012116746
	Δy		-515	282	-115	3020	2817	-109	
	Δz		-377	101	-137	2158	2004	-13	
0-50	Δx		-1425	217	-604			-1036	-.012116746
	Δy		-1177	1524	176			-596	
	Δz		-917	478	-218			-65	
10-30	Δx	47	-32	65	17	705	669		-.010659765
	Δy	119	-21	113	46	-590	-447		
	Δz	36	-8.2	358	14	-215	-184		
30-50	Δx	97	702	64	383	900	875		.093362437
	Δy	393	977	-354	361	-592	-22		
	Δz	93	284	-107	88	-221	-72		

Table II. Moon Reference From Earth to Moon
α = .987883194 Corresponds to Center of Rotation at the Moon

		A	B	C	D	E	F	α
59-60	Δx	-4.8	4.2	-3.5	.38	-1.1	.77	.74542049
	Δy	-17	13	-13	.20	-1.2	.69	
	Δz	-5.6	4.3	-4.2	.02	-1.3	.08	
59-66	Δx	-179	297	-160	67	-219	485	.95560202
	Δy	-849	842	-811	16	-705	648	
	Δz	-278	267	-268	.45	-119	63	
59-71	Δx	-213	838	-211	447		13728	.98774102
	Δy	-2018	1982	-2018	-129		-3695	
	Δz	-715	625	-715	-118		-6390	
66-71	Δx	-13	58	-12	24	-2390	6815	.98774026
	Δy	-409	368	-409	-24	-2795	-8514	
	Δz	-151	130	-151	-12	-395	-5567	

Table IV. Earth Reference Moon Towards Earth

α = -.012116806 Corresponds to Center of Rotation at the Earth

		A	B	C	D	E	F	K	α
85-86	Δx	.453	.285	.174		-.241	-.235	7.26	.63122346
	Δy	-.04	6.46	-283		-5.02	-5.02	-2.00	
	Δz	-.199	3.81	-4.071		.138	.136	-2.52	
85-100	Δx	536	395	704	540	-321	-304	1036	.53630503
	Δy	-207	1817	-6132	-2157	221	208	-291	
	Δz	-246	530	-1020	244	274	269	-366	
85-153	Δx	2268	1486	2268	1894	-667	-309	2317	-.012113913
	Δy	-3553	-14204	-5556	-9812	19431	13105	-4320	
	Δz	430	4819	431	2210	-771	-451	-178	
100-120	Δx	167	168	342	255	-102	-55		.044721505
	Δy	-57	68	-601	-266	158	92		
	Δz	-90	-31	-260	-145	99	82		
100-153	Δx	477	520	477	501				-.012114226
	Δy	-1027	-4630	-1028	-2808				
	Δz	-26	954	-26	424				
120-153	Δx	-11	90	-11	40	40	159		-.12114239
	Δy	60	-962	60	-449	1034	1224		
	Δz	-36	169	-36	63	-206	492		

Table V. Lunar Trajectory - Position Relative to Reference Body

Time in hours	X in Km	Y in Km	Z in Km	Distance in Earth Radii	Reference Body
0	47	6300	1800	1.0	Earth
1	-19000	-8000	-10000	3.6	Earth
10	-45000	-100000	-46000	18.6	Earth
30	-53000	-210000	-82000	36.8	Earth
50	-51000	-290000	-103000	48.8	Earth
59	50000	22000	482	8.6	Moon
60	46000	20000	187	7.96	Moon
66	24000	7300	-1500	3.97	Moon
71	1300	-3700	-2100	.70	Moon
72	-5000	-3500	-681	.96	Moon
73	-10000	-2100	1080	1.62	Moon
75	-19000	1100	4450	3.0	Moon
80	-38000	9100	12000	6.4	Moon
84	-52000	15000	18000	9.0	Moon
85	-57000	-329000	-97000	54.5	Earth
86	-56000	-327000	-95000	54.2	Earth
100	-50000	-300000	-75000	48.9	Earth
120	-36000	-240000	-41000	38.4	Earth
153	2300	-13000	16000	3.3	Earth

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Fourth Semiannual Report

APPLICATION OF VARIATION OF PARAMETERS TO THE POLAR OBLATENESS PROBLEM

RAC 720-7

Research Regarding
Guidance and Space Flight Theory
Relative to the Rendezvous Problem
Contract No. NAS 8-2605

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SUMMARY

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ABST

This report presents the derivation of a set of two body parameters and their associated perturbation equations. These equations are applied to the polar oblateness problem characterized by the second spherical harmonic. A modified Poisson method is used to obtain the first order solution to the problem. The modification of the method is introduced in order to eliminate the occurrence of secular terms which, because of the parameters employed, would have caused a rapid deterioration of the solution. The approximate solution is expressed as a function to true anomaly. Some analysis of second order theory is presented which suggests that difficulties with particular initial conditions may be avoided.

Author

DEFINITION OF SYMBOLS

t	Time
f	True anomaly
$\underline{R}$	Position vector
r	$\lvert\underline{R}\rvert$ = magnitude of $\underline{R}$
μ	Gravitational constant
$\underline{G}$	Angular momentum vector
$\underline{P}$	Eccentricity vector
$\underline{Q}$	$\underline{G} \times \underline{P}$
$\underline{i}$	Unit vector in direction of x axis
$\underline{j}$	Unit vector in direction of y axis
$\underline{k}$	Unit vector in direction of z axis
e	Eccentricity
g	$\lvert\underline{G}\rvert$
p	$\lvert\underline{P}\rvert$
q	$\lvert\underline{Q}\rvert$
σ	Time of perigee passage
a	Semimajor axis
n	Mean motion
K_2	Coefficient of second harmonic of the potential due to the oblateness of the earth
A	$\dfrac{3\mu^2 K_2}{g^4}$

B	$\left(\frac{P_3}{p}\right)^2 + \left(\frac{Q_3}{q}\right)^2$
(r, θ)	Polar coordinate system introduced in x-y plane

SUBSCRIPTS

1,2,3	1st, 2nd, 3rd component of a vector
o	Initial value
s	Short periodic
ℓ	Long periodic

SUPERSCRIPTS

•	Differentiation with respect to time
′	Differentiation with respect to true anomaly

INTRODUCTION

Among the numerous troublesome aspects which one encounters in attempting to integrate the perturbation equations for the polar oblateness problem, two difficulties may occur which appear to be subject to, at least some amelioration. In general, there are two decisions one must make before these difficulties become apparent. These decisions consist of selecting a set of parameters and a method of integrating the perturbation equations. The possible sets of two-body parameters may be divided into two groups, one of which contains canonical parameters and one which does not. Two methods of integration, in general use, are Poisson's method (1) and Von Zeipel's method (2). The latter method is applied only to canonical parameters. In most instances, regardless of the set of two-body parameters or method of integration employed, the results present two interesting properties. The first is the occurrence of terms in the approximate solution which show a secular growth. The second is the presence of singularities in the second order corrections for certain initial conditions of the parameters. The first property is not, in general, objectionable since the secular terms usually appear in the expressions for angle parameters. However, for some parameters, such as the unit perigee vector, the occurrence of secular terms destroys the unit characteristic and limits the applicability of the results to relatively short time intervals.

It is proposed in this report to derive a set of parameters and their associated perturbation equations which, when applied to the polar oblateness problem, yield, after approximate integration, equations for the parameters which manifest no secular growth to the first order, except for one element. A brief analysis of the structure of the second order perturbation equations is developed which suggests that the occurrence of singularities arising from initial conditions is not a necessary concomitant of the polar oblateness problem. The application of second order theory, however, will not be attempted in this report, because the parameters which have been chosen degenerate for nearly circular orbits. Even though the set of parameters employed is defective, the comparative simplicity of the perturbation equations recommends the use of these parameters for a clearer insight into the particular difficulties which their use is intended to eliminate. It should be noted that the degeneracy of the parameters for nearly circular orbits is not a case of replacing one difficulty with another, but is simply a consequence of the choice of parameters and not of the integration technique. A more judicious choice of parameters has been made and an improved integration technique developed which eliminates the imperfections in the present method. A report is now in preparation which incorporates these developments.

DERIVATION OF A SET OF PARAMETERS FOR THE KEPLER PROBLEM

To specify the solution of the vector equation

$$\ddot{\underline{R}} + \frac{\mu \underline{R}}{r^3} = 0 \tag{1}$$

six independent parameters are needed. For the purposes of this report, the following set will be used:

- σ, the time of perigee passage;
- $\underline{P}$, the eccentricity vector;
- $\underline{Q}$, a vector perpendicular to $\underline{P}$ and lying in the plane of motion.

At first glance it would appear that this set contains seven independent elements; but, since $\underline{P}$ and $\underline{Q}$ are mutually orthogonal, any one component may be expressed as a function of the remaining five. The vectors $\underline{P}$ and $\underline{Q}$ may be obtained from Eq. (1) in the following manner: Take the cross product of $\underline{R}$ and Eq. (1)

$$\underline{R} \times \ddot{\underline{R}} = 0 \tag{2}$$

Integration of Eq. (2) gives

$$\underline{R} \times \dot{\underline{R}} = \underline{G} \tag{3}$$

in which $\underline{G}$ is the constant angular momentum vector. Now take the cross product of Eq. (1) and $\underline{G}$

$$\ddot{\underline{R}} \times \underline{G} + \frac{\mu \underline{R}}{r^3} \times \underline{G} = 0 \tag{4}$$

After expanding $\underline{R} \times \underline{G}$ using Eq. (3) and recalling that $\underline{G}$ is constant, Eq. (4) integrates to

$$\dot{\underline{R}} \times \underline{G} - \frac{\mu \underline{R}}{r} = \underline{P} \tag{5}$$

in which $\underline{P}$ is a constant vector. To find the magnitude and direction of $\underline{P}$ rewrite Eq. (5) in the form

$$\underline{P} = \underline{R}\left(\dot{\underline{R}} \cdot \dot{\underline{R}} - \frac{\mu}{r}\right) - \dot{\underline{R}}\,(\underline{R} \cdot \dot{\underline{R}}) \tag{6}$$

Evaluating Eq. (6) at perigee yields

$$\underline{P} = \underline{U}_p\, \mu e \tag{7}$$

where

e is the eccentricity of the orbit

and $\underline{U}_p$ is a unit vector in the direction of perigee. Let $\underline{Q}$ be defined by

$$\underline{Q} = \underline{G} \times \underline{P} = \frac{\mu}{r}\,\underline{R} \times \underline{G} + \dot{\underline{R}}\, g^2 \tag{8}$$

The magnitudes of $\underline{G}$, $\underline{P}$, and $\underline{Q}$ are g, $p = \mu e$, and $q = gp$, respectively.

Since $\underline{R}$, $\underline{P}$, and $\underline{Q}$ are coplanar, $\underline{R}$ may be expressed as a linear combination of $\underline{P}$ and $\underline{Q}$

$$\underline{R} = \alpha_1\, \underline{P} + \alpha_2\, \underline{Q} \tag{9}$$

The scalar product of Eq. (9) with $\underline{P}$ yields

$$\alpha_1 = \frac{\underline{R} \cdot \underline{P}}{p^2} = \frac{r \cos f}{p} \tag{10}$$

where f is the true anomaly of $\underline{R}$. Similarly,

$$\alpha_2 = \frac{\underline{R} \cdot \underline{Q}}{q^2} = \frac{r \sin f}{q} \tag{11}$$

$\dot{\underline{R}}$ may be written as

$$\dot{\underline{R}} = \dot{\alpha}_1\, \underline{P} + \dot{\alpha}_2\, \underline{Q} \tag{12}$$

Making use of the well known formulas

$$r = \frac{g^2}{\mu\,(1 + e \cos f)} \tag{13}$$

$$\dot{f} = \frac{g}{r^2} \tag{14}$$

it follows that

$$\dot{\alpha}_1 = -\frac{\mu \sin f}{g p} \tag{15}$$

$$\dot{\alpha}_2 = \frac{e + \cos f}{g q}\,\mu \tag{16}$$

PERTURBATION EQUATIONS

After having obtained a set of parameters the first step in deriving the perturbation equations is to introduce the perturbing force $\underline{F}$ on the R.H.S. of Eq. (1) which gives

$$\ddot{\underline{R}} + \frac{\mu \underline{R}}{r^3} = \underline{F} \tag{17}$$

The perturbing force $\underline{F}$ will cause $\underline{R}$ to deviate from the Keplerian orbit, and a new solution must be found. This solution can also be put in the form of Eq. (9), but now the parameters $\underline{G}$, $\underline{P}$ and $\underline{Q}$ will be functions of time. In order to determine the time dependences, it will be necessary to obtain the differential equations for the parameters in so far as they depend on the perturbing force $\underline{F}$.

Differentiation of Eq. (3) gives

$$\dot{\underline{G}} = \underline{R} \times \ddot{\underline{R}} \tag{18}$$

Substitution of Eq. (17) yields

$$\dot{\underline{G}} = \underline{R} \times \underline{F} \tag{19}$$

Similarly, differentiation of Eq. (5) gives

$$\dot{\underline{P}} = \ddot{\underline{R}} \times \underline{G} + \dot{\underline{R}} \times \dot{\underline{G}} + \mu \frac{\underline{R} \times \underline{G}}{r^3} \tag{20}$$

Substituting for $\dot{\underline{G}}$ and $\ddot{\underline{R}}$ yields

$$\dot{\underline{P}} = \underline{F} \times \underline{G} + \dot{\underline{R}} \times (\underline{R} \times \underline{F}) \qquad (21)$$

From Eqs. (8), (19) and (21), $\dot{\underline{Q}}$ is given by

$$\dot{\underline{Q}} = \frac{\mu}{r} \underline{R} \times (\underline{R} \times \underline{F}) + \underline{F} g^2 + 2 \dot{\underline{R}} (\underline{G} \cdot \dot{\underline{G}}) \qquad (22)$$

The equation for the variation of σ, the time of perigee passage, is derived from Kepler's equation, which, for $0 < e < 1$, takes the form

$$n (t - \sigma) = \tan^{-1} \sin f \frac{\sqrt{1 - e^2}}{e + \cos f} - \sin f \frac{e\sqrt{1 - e^2}}{1 + e \cos f} \qquad (23)$$

where $n = \sqrt{\frac{\mu}{+a^3}}$ and $g = \sqrt{+ \mu a (1-e^2)}$.

For $e > 1$, Kepler's equation is given by

$$n (t - \sigma) = \tanh^{-1} \sin f \frac{\sqrt{e^2 - 1}}{e + \cos f} - \sin f \frac{e\sqrt{e^2 - 1}}{1 + e \cos f} \qquad (23')$$

where $n = \sqrt{\frac{\mu}{-a^3}}$ and $g = \sqrt{-\mu a(e^2 - 1)}$. Using various identities, Eqs. (23) may be put in the following form

$$n (t - \sigma) = \tan^{-1} \frac{\underline{R} \cdot \dot{\underline{R}}}{(1 - \frac{r}{a}) a^2 n} - \frac{\underline{R} \cdot \dot{\underline{R}}}{a^2 n} \qquad (24)$$

$$n (t - \sigma) = \tanh^{-1} \frac{\underline{R} \cdot \dot{\underline{R}}}{(1 - \frac{r}{a}) a^2 n} - \frac{\underline{R} \cdot \dot{\underline{R}}}{a^2 n} \qquad (24')$$

Differentiation of these equations with respect to time, and substitution of Eq. (17) for $\ddot{\underline{R}}$ gives, in either case

$$\dot{\sigma} = \underline{F} \cdot \left\{ - \frac{3a}{\mu} \dot{\underline{R}} (t - \sigma) + \frac{a}{\mu} \underline{R} + \frac{a^2}{p^2} \left[(1 - e^2) (\dot{\underline{R}} \cdot \underline{R}) \dot{\underline{R}} - \frac{r}{a} \underline{P} \right] \right\} \qquad (25)$$

where

$$-\frac{1}{a} = \frac{\dot{\underline{R}} \cdot \dot{\underline{R}}}{\mu} - \frac{2}{r}$$

and

$$\dot{a} = \dot{\underline{R}} \cdot \underline{F} \left(\frac{2a^2}{\mu}\right)$$

It is convenient to have available the total time derivative of true anomaly. Differentiating the expression

$$\cos f = \frac{\underline{\underline{R}}}{r} \cdot \frac{\underline{\underline{P}}}{p} \tag{26}$$

it follows that

$$-(\sin f)\,\dot{f} = \frac{r\,\dot{\underline{R}} - \dot{\underline{R}}\,\dot{r}}{r^2} \cdot \frac{\underline{\underline{P}}}{p} + \frac{\underline{\underline{R}}}{r} \cdot \left(\frac{\underline{\underline{P}}}{p}\right) \tag{27}$$

and therefore

$$\dot{f} = \frac{g}{r^2} - \frac{\underline{\underline{P}}}{p} \cdot \frac{\underline{\underline{Q}}}{q} \tag{28}$$

APPLICATION OF THE PERTURBATION EQUATIONS TO THE POLAR OBLATENESS PROBLEM

In this report, the polar oblateness problem will be assumed to be characterized by the perturbing potential

$$\Phi = \frac{\mu K_2}{r^3}\left(1 - 3\,\frac{z^2}{r^2}\right) \tag{29}$$

In order to apply the perturbation equations, previously presented, to this problem, it is necessary to specify the perturbing force $\underline{F}$. This force is the gradient of the perturbing potential Φ.

$$\underline{F} = -\frac{3\mu K_2}{r^5}\left\{\left[1 - 5\frac{z^2}{r^2}\right]\underline{R} + 2z\,\underline{k}\right\} \tag{30}$$

The procedure for applying the perturbation equations may be outlined as follows:

(a) Reexpress the perturbation equations in terms of the parameters $\underline{P}$, $\underline{Q}$, and $\underline{G}$, and true anomaly, f, by substituting Eqs. (9), (12), and (30) for $\underline{R}$, $\dot{\underline{R}}$, and $\underline{F}$, respectively.

(b) Since the resulting equations are functions of true anomaly, it is legitimate to take $\dot{f} = g/r^2$, for a first order approximation. It follows that the differential equations with respect to time may be transformed to differential equations with respect to true anomaly.

(c) These perturbation equations are now written as Fourier polynomials. Terms with constant coefficients are transposed to the L. H. S.

(d) To obtain a first order solution for the system of equations derived in (c), all parameters on the R. H. S. and the parameter g, wherever it occurs, are held constant. Under these conditions, the system can be solved exactly.

(e) The perturbation equation for the parameter σ is treated similarly with some modifications.

Carrying out the operations indicated in (a), (b), and (c) the results are:

$$\begin{aligned}
\underline{P}' - \frac{3\mu^3 K_2}{g^4}\, e\left\{\underline{k}\,\frac{Q_3}{q} - \frac{\underline{P}}{p}\,\frac{P_3 Q_3}{p\,q} - \frac{\underline{Q}}{q}\left[\frac{3}{2}\,\frac{P_3^{\,2}}{p^2} + \frac{5}{2}\,\frac{Q_3^{\,2}}{q^2} - 1\right]\right\} \\
= \frac{3\mu K_2}{g^4}\left\{\underline{k}\left[\frac{P_3}{p}\left(\frac{e^2}{2}\sin 3f + e\sin 2f + \frac{e^2}{2}\sin f\right) - \frac{Q_3}{q}\left(\frac{e^2}{2}\cos 3f\right.\right.\right. \\
\left.\left. + e\cos 2f - \frac{e^2}{2}\cos f\right)\right] + \frac{\underline{P}}{p}\left[\left(\frac{P_3}{p}\right)^2\left(\frac{5e^2}{16}\sin 5f + \frac{3e}{2}\sin 4f\right.\right. \\
\left. + \left(\frac{7}{4} + \frac{15e^2}{16}\right)\sin 3f + 3e\sin 2f + \left(\frac{7}{4} + \frac{5e^2}{8}\right)\sin f\right) - \frac{P_3 Q_3}{p\,q}\left(\frac{5e^2}{8}\cos 5f\right. \\
\left. + 3e\cos 4f + \left(\frac{7}{2} + \frac{13}{8}e^2\right)\cos 3f + 4e\cos 2f + \left(\frac{1}{2} + \frac{7}{4}e^2\right)\cos f\right)
\end{aligned} \tag{31}$$

$$- \left(\frac{Q_3}{q}\right)^2 \left(\frac{5e^2}{16} \sin 5 f + \frac{3e}{2} \sin 4 f + (\frac{7}{4} + \frac{11 e^2}{16}) \sin 3 f + e \sin 2 f\right.$$

$$\left. + (\frac{3e^2}{8} - \frac{5}{4}) \sin f\right) - \left(\frac{e^2}{4} \sin 3 f + e \sin 2 f + (1 + \frac{e^2}{4}) \sin f\right)\Big]$$

$$+ \frac{Q}{q} \Big[- \left(\frac{P_3}{p}\right)^2 \left(\frac{5e^2}{16} \cos 5 f + \frac{3e}{2} \cos 4 f + (\frac{7}{4} + \frac{17}{16} e^2) \cos 3 f + 3e \cos 2 f\right.$$

$$\left. + (\frac{5}{4} + \frac{13}{8} e^2) \cos f\right) + \left(\frac{Q_3}{q}\right)^2 \left(\frac{5e^2}{16} \cos 5 f + \frac{3e}{2} \cos 4 f + (\frac{7}{4} + \frac{13}{16} e^2) \cos 3 f\right.$$

$$\left. + e \cos 2 f - (\frac{7}{4} + \frac{9e^2}{8} \cos f\right) - \frac{P_3 Q_3}{p\ q} \left(\frac{5e^2}{2} \sin 5f + 3e \sin 4f + (\frac{7}{2} + 3e^2) \sin 3f\right.$$

$$\left. + 4e \sin 2f + (\frac{e^2}{2} - \frac{1}{2}) \sin f\right) + \left(\frac{e^2}{4} \cos 3 f + e \cos 2 f + (1 + \frac{3e^2}{4}) \cos f\right)\Big]\Big\}$$

(31) cont'd

$$\underline{Q}' - \frac{3\mu^3 K_2 e}{g^3} \left\{ - \underline{k} \frac{P_3}{p} + \frac{P}{p}\left[\frac{5}{2} \frac{P_3^2}{p^2} + \frac{3}{2} \frac{Q_3^2}{q^2} - 1 \right] + \frac{Q}{q} \frac{P_3}{p} \frac{Q_3}{q} \right\}$$

$$= \frac{3\mu^3 K_2}{g^3} \left\{ - \underline{k} \Big[\frac{P_3}{p} \left(\frac{e^2}{2} \cos 3 f + e \cos 2 f + \frac{3e^2}{2} \cos f\right) + \frac{Q_3}{q} \left(\frac{e^2}{2} \sin 3 f\right.\right.$$

$$\left. + e \sin 2 f + \frac{e^2}{2} \sin f\right)\Big] + \frac{P}{p} \Big[\left(\frac{P_3}{p}\right)^2 \left(\frac{5e^2}{16} \cos 5f + \frac{3e}{2} \cos 4 f\right.$$

$$\left. + (\frac{7}{4} + \frac{25}{16} e^2) \cos 3f + 4e \cos 2f + (\frac{5}{4} + \frac{25}{8} e^2) \cos f\right) - \left(\frac{Q_3}{q}\right)^2 \left(\frac{5e^2}{16} \cos 5f\right.$$

$$+ \frac{3e}{2} \cos 4f + (\frac{7}{4} + \frac{5e^2}{16}) \cos 3f - \left.(\frac{7}{4} + \frac{5}{8} e^2) \cos f\right) + \frac{P_3 Q_3}{p\ q} \left(\frac{5e^2}{2} \sin 5 f\right.$$

$$+ 3e \sin 4f + (\tfrac{7}{2} + 3e^2) \sin 3f + 4e \sin 2f + (\tfrac{e^2}{2} - \tfrac{1}{2}) \sin f\Big) - \Big(\frac{e^2}{4} \cos 3f$$

$$+ e \cos 2f + (1 + \frac{3e^2}{4}) \cos f\Big)\Big] + \frac{Q}{q}\Big[\Big(\frac{P_3}{p}\Big)^2 \Big(\frac{5e^2}{16} \sin 5f + \frac{3e}{2} \sin 4f$$

$$+ (\tfrac{7}{4} + \tfrac{31}{16} e^2) \sin 3f + 5e \sin 2f + (\tfrac{7}{4} + \frac{13e^2}{8}) \sin f\Big) - \Big(\frac{Q_3}{q}\Big)^2 \Big(\frac{5e^2}{16} \sin 5f$$

$$+ \frac{3e}{2} \sin 4f + (\tfrac{7}{4} + \frac{11e^2}{16}) \sin 3f + e \sin 2f + (\frac{3e^2}{8} - \tfrac{5}{4}) \sin f\Big)$$

$$- \frac{P_3 Q_3}{p\ q}\Big(\frac{5e^2}{8} \cos 5f + 3e \cos 4f + (\tfrac{7}{2} + \tfrac{21}{8} e^2) \cos 3f + 6e \cos 2f$$

$$+ (\tfrac{1}{2} + \frac{3e^2}{4}) \cos f\Big) - \Big(\frac{e^2}{4} \sin 3f + e \sin 2f + (1 + \frac{3e^2}{4}) \sin f\Big)\Big]\Big\} \qquad \text{(31) cont'd}$$

$$\underline{G}' + \frac{3\mu^2 K_2}{g^3} \Big\{\frac{\underline{P}}{p} \frac{P_3}{p} + \frac{\underline{Q}}{q} \frac{Q_3}{q}\Big\} \times \underline{k}$$

$$= \frac{3\mu^2 K_2}{g^3} \Big\{\frac{\underline{P}}{p}\Big[\frac{P_3}{p}\Big(\frac{e}{2} \cos 3f + \cos 2f + \frac{3e}{2} \cos f\Big) + \frac{Q_3}{q}\Big(\frac{e}{2} \sin 3f + \sin 2f$$

$$+ \frac{e}{2} \sin f\Big)\Big] + \frac{\underline{Q}}{q}\Big[\frac{P_3}{p}\Big(\frac{e}{2} \sin 3f + \sin 2f + \frac{e}{2} \sin f\Big) + \frac{Q_3}{q}\Big(-\frac{e}{2} \cos 3f - \cos 2f$$

$$+ \frac{e}{2} \cos f\Big)\Big]\Big\} \times \underline{k}$$

where

$$(\quad)' = \frac{d(\quad)}{df}$$

Consider the system of homogeneous equations obtained by setting the R. H. S. of Eqs. (31) equal to zero.

$$\underline{P}'_\ell - \frac{3\mu^2 K_2 p_\ell}{g_\ell^{\ 4}} \left\{ \underline{k} \frac{Q_{3\ell}}{q_\ell} - \frac{\underline{P}_\ell}{p_\ell} \frac{P_{3\ell}}{p_\ell} \frac{Q_{3\ell}}{q_\ell} - \frac{\underline{Q}_\ell}{q_\ell} \left[\frac{3}{2} \frac{P_{3\ell}^{\ 2}}{p_\ell^{\ 2}} + \frac{5}{2} Q_{3\ell}^{\ 2} - 1 \right] \right\} = 0$$

$$\underline{Q}'_\ell - \frac{3\mu^2 K_2 q_\ell}{g_\ell^{\ 4}} \left\{ -\underline{k} \frac{P_{3\ell}}{p_\ell} + \frac{\underline{P}_\ell}{p_\ell} \left[\frac{5}{2} \frac{P_{3\ell}^{\ 2}}{p_\ell^{\ 2}} + \frac{3}{2} \frac{Q_{3\ell}^{\ 2}}{q_\ell^{\ 2}} - 1 \right] + \frac{\underline{Q}_\ell}{q_\ell} \frac{P_{3\ell}}{p_\ell} \frac{Q_{3\ell}}{q_\ell} \right\} = 0 \qquad (32)$$

$$\underline{G}'_\ell + \frac{3\mu^2 K_2 g_\ell}{g_\ell^{\ 4}} \left\{ \frac{\underline{P}_\ell}{p_\ell} \frac{P_{3\ell}}{p_\ell} + \frac{\underline{Q}_\ell}{q_\ell} \frac{Q_{3\ell}}{q_\ell} \right\} \times \underline{k} = 0$$

It will become apparent that $\underline{P}_\ell$, $\underline{Q}_\ell$, and $\underline{G}_\ell$ represent the long periodic terms of $\underline{P}$, $\underline{Q}$, and $\underline{G}$, respectively.

For this system of equations, Eq. (8), $\underline{Q}_\ell = \underline{G}_\ell \times \underline{P}_\ell$ still holds. Since

$$p^2_\ell = \underline{P}_\ell \cdot \underline{P}_\ell \qquad q^2_\ell = \underline{Q}_\ell \cdot \underline{Q}_\ell \qquad g^2_\ell = \underline{G}_\ell \cdot \underline{G}_\ell$$

It follows from Eqs. (32) that

$$p'_\ell = \frac{\underline{P}_\ell}{p_\ell} \cdot \underline{P}'_\ell = 0$$

$$q'_\ell = \frac{\underline{Q}_\ell}{q_\ell} \cdot \underline{Q}'_\ell = 0 \qquad (33)$$

$$g'_\ell = \frac{\underline{G}_\ell}{g_\ell} \cdot \underline{G}'_\ell = 0$$

Therefore, for this system of equations, p_ℓ, q_ℓ, and g_ℓ are constant. Similarly,

$$\frac{P_{3\ell}}{p_\ell} \left(\frac{P_{3\ell}}{p_\ell} \right)' + \frac{Q_{3\ell}}{q_\ell} \left(\frac{Q_{3\ell}}{q_\ell} \right)' = 0 \qquad (34)$$

so that

$$\left(\frac{P_{3\ell}}{p_\ell}\right)^2 + \left(\frac{Q_{3\ell}}{q_\ell}\right)^2 = B \tag{35}$$

is constant.

Using the identity

$$\underline{k} \times \frac{\underline{G}_\ell}{g_\ell} = \frac{\underline{P}_\ell}{p_\ell}\,\frac{Q_{3\ell}}{q_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\,\frac{P_{3\ell}}{p_\ell} \tag{36}$$

it follows that

$$\begin{aligned}\left(\frac{\underline{P}_\ell}{p_\ell}\,\frac{P_{3\ell}}{p_\ell} + \frac{\underline{Q}_\ell}{q_\ell}\,\frac{Q_{3\ell}}{q_\ell}\right) \times \underline{k} &= \left(\frac{\underline{Q}_\ell}{q_\ell} \times \frac{\underline{G}_\ell}{g_\ell}\,\frac{P_{3\ell}}{p_\ell} + \frac{\underline{G}_\ell}{g_\ell} \times \frac{\underline{P}_\ell}{p_\ell}\,\frac{Q_{3\ell}}{q_\ell}\right) \times \underline{k} \\ &= \frac{G_{3\ell}}{g_\ell}\left[\frac{\underline{P}_\ell}{p_\ell}\,\frac{Q_{3\ell}}{q_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\,\frac{P_{3\ell}}{p_\ell}\right] = \frac{G_{3\ell}}{g_\ell}\,\underline{k} \times \frac{\underline{G}_\ell}{g_\ell}\end{aligned} \tag{37}$$

Therefore, Eqs. (32), can be rewritten as

$$\begin{aligned}&\underline{P}'_\ell + A p_\ell \left\{\frac{P_{3\ell}}{p_\ell}\,\underline{k} \times \frac{\underline{G}_\ell}{g_\ell} - \frac{\underline{Q}_\ell}{q_\ell}\left(1 - \frac{5}{2}B\right) - \underline{k}\,\frac{Q_{3\ell}}{q_\ell}\right\} = 0 \\ &Q'_\ell + A q_\ell \left\{\frac{Q_{3\ell}}{q_\ell}\,\underline{k} \times \frac{\underline{G}_\ell}{g_\ell} - \frac{\underline{P}_\ell}{p_\ell}\left(\frac{5}{2}B - 1\right) + \underline{k}\,\frac{P_{3\ell}}{p_\ell}\right\} = 0 \\ &\underline{G}'_\ell + A g_\ell \left\{\frac{G_{3\ell}}{g_\ell}\,\underline{k} \times \frac{\underline{G}_\ell}{g_\ell}\right\} = 0\end{aligned} \tag{38}$$

where $A = \dfrac{3\mu^2 K_2}{g_\ell^4}$

The third components of $\underline{P}'_\ell$, $\underline{Q}'_\ell$, and $\underline{G}'_\ell$ are

$$P'_{3\ell} + Ap_\ell \frac{Q_{3\ell}}{q_\ell}\left(\frac{5}{2}B - 2\right) = 0$$

$$Q'_{3\ell} + Aq_\ell \frac{P_{3\ell}}{p_\ell}\left(2 - \frac{5}{2}B\right) = 0 \tag{39}$$

$$G'_{3\ell} = 0$$

which form a system of first order, linear, homogeneous differential equations with constant coefficients. The solution is

$$P_{3\ell} = P_{3o} \cos\left\{A\left(\frac{5}{2}B - 2\right)f\right\} - Q_{3o} \sin\left\{A\left(\frac{5}{2}B - 2\right)f\right\}$$

$$Q_{3\ell} = P_{3o} \sin\left\{A\left(\frac{5}{2}B - 2\right)f\right\} + Q_{3o} \cos\left\{A\left(\frac{5}{2}B - 2\right)f\right\} \tag{40}$$

$$G_{3\ell} = G_{3o}$$

where P_{3o}, Q_{3o}, and G_{3o} are initial conditions. Similarly, the first two components of $\underline{G}'$ are

$$G'_{1\ell} - A\,G_{2\ell} \frac{G_{3\ell}}{g_\ell} = 0$$

$$G'_{2\ell} + A\,G_{1\ell} \frac{G_{3\ell}}{g_\ell} = 0 \tag{41}$$

This system has the solution

$$G_{1\ell} = G_{1o} \cos\left(A \frac{G_{3\ell}}{g_\ell} f\right) + G_{2o} \sin\left(A \frac{G_{3\ell}}{g_\ell} f\right)$$

$$G_{2\ell} = -G_{1o} \sin\left(A \frac{G_{3\ell}}{g_\ell} f\right) + G_{2o} \cos\left(A \frac{G_{3\ell}}{g_\ell} f\right) \tag{42}$$

where G_{1o} and G_{2o} are initial conditions. Using the identities,

$$\frac{G_{1\ell}}{g_\ell}\frac{G_{3\ell}}{g_\ell} = -\frac{P_{1\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} - \frac{Q_{1\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell}$$

$$\frac{G_{2\ell}}{g_\ell}\frac{G_{3\ell}}{g_\ell} = -\frac{P_{2\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} - \frac{Q_{2\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell} \tag{43}$$

Eqs. (41) may be transformed into

$$G'_{1\ell} + A\, g_\ell \left\{\frac{P_{2\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{Q_{2\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell}\right\} = 0$$

$$G'_{2\ell} - A\, g_\ell \left\{\frac{P_{1\ell}}{p_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{Q_{1\ell}}{q_\ell}\frac{Q_{3\ell}}{q_\ell}\right\} = 0 \tag{44}$$

Eqs. (44) together with the identities

$$\frac{G_{1\ell}}{g_\ell} = \frac{P_{2\ell}}{p_\ell}\frac{Q_{3\ell}}{q_\ell} - \frac{P_{3\ell}}{p_\ell}\frac{Q_{3\ell}}{q_\ell}$$

$$\frac{G_{2\ell}}{g_\ell} = \frac{P_{3\ell}}{p_\ell}\frac{Q_{1\ell}}{q_\ell} - \frac{P_{1\ell}}{p_\ell}\frac{Q_{3\ell}}{q_\ell} \tag{45}$$

determine the remaining components of $\underline{P}$ and $\underline{Q}$ which are

$$P_{1\ell} = \frac{p_\ell}{AB}\left\{\frac{P_{3\ell}}{p_\ell}\left(\frac{G_{2\ell}}{g_\ell}\right)' - A\frac{G_{2\ell}}{g_\ell}\frac{Q_{3\ell}}{q_\ell}\right\}$$

$$P_{2\ell} = \frac{p_\ell}{AB}\left\{A\frac{G_{1\ell}}{g_\ell}\frac{Q_{3\ell}}{q_\ell} - \frac{P_{3\ell}}{p_\ell}\left(\frac{G_{1\ell}}{g_\ell}\right)'\right\}$$

$$Q_{1\ell} = \frac{q_\ell}{AB}\left\{A\frac{G_{2\ell}}{g_\ell}\frac{P_{3\ell}}{p_\ell} + \frac{Q_{3\ell}}{q_\ell}\left(\frac{G_{2\ell}}{g_\ell}\right)'\right\} \tag{46}$$

$$Q_{2\ell} = \frac{q_\ell}{AB}\left\{-A\frac{G_{1\ell}}{g_\ell}\frac{P_{3\ell}}{p_\ell} - \frac{Q_{3\ell}}{q_\ell}\left(\frac{G_{1\ell}}{g_\ell}\right)'\right\}$$

All quantities appearing on the R.H.S. of Eqs. (46) are known. After some algebraic manipulation, the solution for the system of Eqs. (32) may be expressed as

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ Q_1 \\ Q_2 \\ Q_3 \\ G_1 \\ G_2 \\ G_3 \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix} \cdot \begin{bmatrix} \cos A\left(\frac{5}{2}B-2\right)f\,I & -\sin A\left(\frac{5}{2}B-2\right)f\,I & 0 \\ \sin A\left(\frac{5}{2}B-2\right)f\,I & \cos A\left(\frac{5}{2}B-2\right)f\,I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} P_{1o} \\ P_{2o} \\ P_{3o} \\ Q_{1o} \\ Q_{2o} \\ Q_{3o} \\ G_{1o} \\ G_{2o} \\ G_{3o} \end{bmatrix} \tag{47}$$

where

$$C = \begin{bmatrix} \cos A\,\frac{G_{3\ell}}{g_\ell}\,f & \sin A\,\frac{G_{3\ell}}{g_\ell}\,f & 0 \\ -\sin A\,\frac{G_{3\ell}}{g_\ell}\,f & \cos A\,\frac{G_{3\ell}}{g_\ell}\,f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the particular solution of Eqs. (31), assume a solution of the form (47) where $\underline{P}_o$, $\underline{Q}_o$, and $\underline{G}_o$ are functions of f. Substituting solution (47) into the L.H.S. of Eqs. (31) will yield three equations for $\underline{P}_o'$, $\underline{Q}_o'$, and $\underline{G}_o'$. After solving for these derivatives, and recalling condition (d), $\underline{P}_o$, $\underline{Q}_o$, and $\underline{G}_o$ may then be found by integration alone. If the second order terms in this solution are neglected, the results are equivalent to integrating the R.H.S. of Eqs. (31) and adding the results to solution (47). The first order solution for $\underline{P}$, $\underline{Q}$, $\underline{G}$, is

$$\begin{aligned}\underline{P} &= \underline{P}_\ell + \left[\underline{P}_s\right]_{f_o}^{f} \\ \underline{Q} &= \underline{Q}_\ell + \left[\underline{Q}_s\right]_{f_o}^{f} \\ \underline{G} &= \underline{G}_\ell + \left[\underline{G}_s\right]_{f_o}^{f}\end{aligned} \tag{48}$$

where $\underline{P}_s$, $\underline{Q}_s$, $\underline{G}_s$, are the integrals of the R.H.S. of Eqs. (31), and the quantities in brackets are to be evaluated between the limits f and f_o.

In the perturbation equation for σ, Eq. (25) it may be noted that

$$\underline{R} \cdot \underline{F} = -3\Phi \qquad \dot{\underline{R}} \cdot \underline{F} = \frac{d\Phi}{dt}$$

If the parameters a and σ are held constant at their initial values

$$\frac{d}{dt}\left\{-\frac{3a_o}{\mu}\Phi(t-\sigma_o)\right\} = \left\{-\frac{3a_o}{\mu}\dot{\underline{R}}(t-\sigma_o) + \frac{a_o\underline{R}}{\mu}\right\} \cdot \underline{F} \tag{49}$$

Therefore, Eq. (25) may be rewritten in the form

$$\begin{aligned}&\frac{d}{dt}\left\{\sigma + \frac{3a_o}{\mu}\Phi(t-\sigma_o)\right\} \\ &\quad = \frac{a^2}{p^2}\left\{(1-e^2)(\dot{\underline{R}} \cdot \underline{R})\dot{\underline{R}} - \frac{r}{a}\underline{P}\right\} \cdot \underline{F}\end{aligned} \tag{50}$$

Differentiation with respect to time is transformed to differentiation with respect to true anomaly, and the R.H.S. is expressed as a Fourier polynomial. The result is

$$\frac{d}{dt}\left\{\sigma + \frac{3a_o}{\mu}\Phi(t-\sigma_o)\right\} = \frac{3\mu^2 K_2 a}{g^3 p}\left\{-\left(\frac{P_3}{p}\right)^2\left[\frac{5e^2}{16}\cos 5f\right.\right.$$

$$+\frac{3e}{2}\cos 4f + \left(\frac{5e^2}{16}+\frac{7}{4}\right)\cos 3f + \frac{3e}{2}\cos 2f + \left.\left(\frac{5}{4}-\frac{5e^2}{8}\right)\cos f\right]$$

$$+\left(\frac{Q_3}{q}\right)^2\left[\frac{5e^2}{16}\cos 5f + \frac{3e}{2}\cos 4f + \left(\frac{7}{4}-\frac{7}{16}e^2\right)\cos 3f\right.$$

$$\left.-\frac{3e}{2}\cos 2f + \left(\frac{e^2}{8}-\frac{7}{4}\right)\cos f\right] - \frac{P_3}{p}\frac{Q_3}{q}\left[\frac{5e^2}{8}\sin 5f - \frac{e}{2}\sin 4f\right. \tag{51}$$

$$\left.+\left(\frac{e^2}{8}-\frac{7}{4}\right)\cos f\right] - \frac{P_3}{p}\frac{Q_3}{q}\left[\frac{5e^2}{8}\sin 5f - \frac{e}{2}\sin 4f + \left(\frac{7}{2}-\frac{e^2}{8}\right)\sin 3f\right.$$

$$\left.+ 3e\sin 2f - \left(\frac{3e^2}{4}+\frac{1}{2}\right)\sin f\right] + \left[\frac{e^2}{4}\cos 3f + e\cos 2f\right.$$

$$\left.\left.+\left(1-\frac{e^2}{4}\right)\cos f\right]\right\}$$

Holding the parameters on the R.H.S. constant, Eq. (51) is integrated to yield

$$\sigma = \sigma_o + \left[\sigma_s - \frac{3a_o\Phi}{\mu}(t-\sigma_o)\right]_{f_o}^{f} \tag{52}$$

where σ_s is the integral of the second member of Eq. (51).

CONCLUSION

The solution (47) obtained has f appearing in arguments of sines and cosines, these terms having two essentially different periods: $2\pi/j$ (short period where j is a natural number), and $2\pi/A$ (long period where A is a small quantity and equals $3\mu^2 K_2/g^4$). The solution is well behaved for all values of f because f appears in arguments of sines and cosines and because these functions are found only in the numerator. This would not be the case if Eqs. (31) were integrated keeping all parameters constant; for then, the long periodic terms in the previous solution would be replaced by their first order approximations. This solution would grow linearly with time.

The next step in the usual procedure for deriving the second order approximation consists in substituting the first order solution for the parameters in Eqs. (31). Before this step can be carried out, however, it should be recalled that Eqs. (31) were obtained by putting $dt/df = r^2/g$. If higher order solutions are to be found, this approximation is no longer valid. Therefore, for a second order approximation, dt/df must be replaced by its first order approximation derived from Eq. (28).

Now suppose the parameters are replaced by their first order solutions, terms of order $K_2{}^3$ are neglected, and products of trigonometric functions are replaced by trigonometric functions of sums. Under the following conditions, the resulting equations may be integrated to give a well behaved second order solution:

(a) No constant terms are present

(b) Whenever $\cos \alpha f$ or $\sin \alpha f$ occurs (α a small quantity), α must also appear as a factor in the numerator.

If these conditions are not fulfilled, and the equations are integrated, f may occur outside trigonometric functions, or small divisors may be present. A possible solution to these difficulties is obtained as follows:

(a) Denote the short periodic terms of the first order solution of $\underline{P}$, $\underline{Q}$, $\underline{G}$ by $\underline{P}_s(\underline{P}_0, \underline{Q}_0, f)$, $\underline{Q}_s(\underline{P}_0, \underline{Q}_0, f)$, $\underline{G}_s(\underline{P}_0, \underline{Q}_0, f)$ and assume a solution of the form $\underline{P} = \underline{P}_\ell + \underline{P}_s(\underline{P}_\ell, \underline{Q}_\ell, f)$, $\underline{Q} = \underline{Q}_\ell + \underline{Q}_s(\underline{P}_\ell, \underline{Q}_\ell, f)$, $\underline{G} = \underline{G}_\ell + \underline{G}_s(\underline{P}_\ell, \underline{Q}_\ell, f)$ $\underline{P}_\ell$, $\underline{Q}_\ell$, $\underline{G}_\ell$ are new variables which, to first order, are equivalent to solution (47).

(b) Substitute these expressions into both sides of Eqs. (31) as modified in accordance with the qualification regarding dt/df mentioned above. Neglect terms of order $K_2{}^3$; expand into Fourier polynomials, and neglect terms multiplied by sines and cosines. $\underline{P}_\ell$, $\underline{Q}_\ell$, $\underline{G}_\ell$ are determined from the resulting equations.

Investigations are currently being pursued for the purpose of finding the second order solution by this method.

APPENDIX

EXAMPLE OF RAPIDLY VARYING PARAMETERS

Whenever perturbation equations for a set of parameters are solved employing an approximate integration method, it is always desirable that the parameters be slowly varying. It is likely that, for the polar oblateness problem, no set of parameters exist in which all elements possess this characteristic. An example is presented to demonstrate the existence of rapidly varying parameters for the polar oblateness problem. Consider the equation

$$\ddot{z} + \frac{\mu z}{r^3} = -\frac{3\mu K_2}{r^5}\left[\left(1 - 5\frac{z^2}{r^2}\right)z + 2\,z\right]$$

which is obtained by taking the scalar product of Eq. (30) with $\underline{k}$. Given the initial conditions $z\,(t_o) = \dot{z}\,(t_o) = 0$, it follows that all derivatives of z evaluated at $t - t_o$ are zero. Therefore z is identically zero.

In the following example it is to be assumed that this is the case. Then $\dot{\underline{G}} = \underline{R} \times \underline{F} = 0$ or $\underline{G} = G_3\,\underline{k}$ where G_3 is a constant. Now introduce a polar coordinate system, (r, θ) in the x-y plane. From Eq. (30) two scalar equations result:

$$\ddot{r} - r\,(\dot{\theta})^2 = -\frac{\mu}{r^2} - \frac{3\mu K_2}{r^4}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\,\dot{\theta}) = 0$$

A particular solution of these equations is given by

$$r = r_o, \; \theta = \theta_o \pm t\sqrt{\frac{\mu}{r_o^3} + \frac{3\mu K_2}{r_o^5}}$$

where r_o, θ_o are constant. Since

$$g = |r^2 \dot{\theta}| = \sqrt{r_o \mu + \frac{3\mu K_2}{r_o}}$$

and

$$e \cos f = \frac{g^2}{\mu r_o} - 1$$

it follows that

$$e \cos f = \frac{3 K_2}{r_o^2}$$

Also, $\dot{r} = 0$, so that

$$\underline{R} \;\; \dot{\underline{R}} = r\,\dot{r} = \frac{\mu r_o e \sin f}{g} = 0$$

As a result it is seen that $e \sin f = 0$. Therefore, it may be concluded that $e > 0$, $f \equiv 0$. From the equation

$$\underline{R} = r\left(\cos f \frac{\underline{P}}{p} + \sin f \frac{\underline{Q}}{q}\right)$$

one obtains

$$\underline{R} = r \frac{\underline{P}}{p}$$

It is clear that the vector $\underline{P}$ is always in the direction of the vector $\underline{R}$ and is thus a rapidly varying parameter. Consequently, there is no guarantee that the method of variation of parameters and an approximate integration procedure will yield a satisfactory solution.

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APPROXIMATE INITIAL VALUES OF LAGRANGE MULTIPLIERS FOR THE TWO POINT BOUNDARY VALUE PROGRAM

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ABST

This report describes a method for obtaining a first estimate of initial values of the Lagrange multipliers for the "two point boundary value problem of the calculus of variations."

This first estimate is obtained by assuming the "two impulse orbit transfer" problem to be a reasonably close approximation to the calculus of variations problem.

Author

DEFINITION OF SYMBOLS

μ	Gravitational constant
$\underline{R}$	Vehicle position vector
r	$\lvert \underline{R} \rvert$ = magnitude of $\underline{R}$
$\underline{V}$	Velocity vector of vehicle
$\Delta \underline{V}$	Impulse velocity vector
ΔV	$\lvert \Delta \underline{V} \rvert$ = magnitude of $\Delta \underline{V}$
k	Magnitude of thrust
$\underline{T}$	Unit vector in direction of thrust
m	Mass of vehicle
$\dot{m}$	Mass flow
c	Constant, proportional to specific impulse
$\underline{\lambda}$, $\dot{\underline{\lambda}}$, σ	Lagrange multipliers or adjoint variables
λ	$\lvert \underline{\lambda} \rvert$ = magnitude of $\underline{\lambda}$
λ	$\lvert \dot{\underline{\lambda}} \rvert$ = magnitude of $\dot{\underline{\lambda}}$
λ_{ξ}	Component of $\underline{\lambda}$ parallel to $\underline{R}$
λ_{η}	Component of $\underline{\lambda}$ perpendicular to $\underline{R}$
t	Time
t_1	Time at end of first thrust period
t_2	Time at beginning of second thrust period

SUBSCRIPTS

o	**Initial value**
f	**Final value**

INTRODUCTION

The method used to solve the two point boundary value problem of the calculus of variations is one where the decision functions are such that all the trajectories being used are extremals [1]. In addition to the state variables, that appear in the equations of motion, there are a number of adjoint variables or Lagrange multipliers that satisfy additional equations for the optimization of the given system. The boundary conditions for the adjoint variables define the natural end-point conditions of the state variables. This natural end point, in general will not be the desired end point. A differential correction scheme provide the means of obtaining another optimum trajectory, the natural end point of which will be closer to the desired end point [2].

The equations of motion of the vehicle in the gravitational field of a single body subject to thrust are as follows:

$$\ddot{\underline{R}} = -\frac{\mu \underline{R}}{r^3} + \frac{k}{m} \underline{T} \qquad (1)$$

$$m\,(t_B) = m\,(t_A) + \int_{t_A}^{t_B} \dot{m}\, dt \qquad (2)$$

where $\dot{m} = -\frac{k}{c}$ and $\underline{T}$ is a unit vector parallel to the direction of thrust.

The optimum decision functions are determined with the help of the Lagrange multipliers, $\underline{\lambda}$, $\dot{\underline{\lambda}}$, and σ which satisfy the following equations

$$\ddot{\underline{\lambda}} = -\frac{\mu \underline{\lambda}}{r^3} + \frac{3\mu(\underline{\lambda} \quad \underline{R})\,\underline{R}}{r^5} \qquad (3)$$

$$\sigma\,(t_B) = \sigma\,(t_A) + \int_{t_A}^{t_B} \dot{\sigma}\, dt \qquad (4)$$

where

$$\dot{\sigma} = \frac{k\,\lambda}{m^2} .$$

The thrusting program is determined by the sign of the switching function S, which is given by

$$S = \left(\frac{\lambda}{m} - \frac{\sigma}{c} \right) \qquad \begin{matrix} > 0 & k = k_{max} \\ < 0 & k = k_{min} \end{matrix} \tag{5}$$

The direction of the unit thrust vector $\underline{T}$ is given by the direction of the Lagrange multiplier $\underline{\lambda}$

$$\underline{T} = \frac{\underline{\lambda}}{\lambda} \tag{6}$$

The natural end point if reached when

$$\sigma\,(t_F) = 1 \tag{7}$$

The problem is to generate a set of initial values of the Lagrange multipliers such that an optimum orbit can be computed, where the natural end point matches the desired end point. This is accomplished by obtaining a first estimate of the initial values and improving these by using a differential correction scheme.

One of the requirements necessary for a rapid convergence of the differential correction scheme is that the first estimate of the initial values of the Lagrange multipliers be reasonably close. The following is a method for obtaining a first crude estimate of the initial values of the Lagrange multipliers.

INITIAL VALUES OF LAGRANGE MULTIPLIERS

First Method

A first estimate for the initial values of the Lagrange multipliers can be obtained by making the following assumptions about the trajectory.

(a) Two burning periods are required to accomplish the optimum trajectory, one occurring in the time interval t_o to t_1 and the other in the time interval t_2 to t_f. During the time interval t_1 to t_2 the vehicle is in a coasting region.

(b) The time intervals in the thrust regions are so small that $\Delta \underline{V}(t_o)$ and $\Delta \underline{V}(t_F)$ are obtained by solving the "two-impulse orbit transfer" problem, where

$$\begin{aligned} \Delta \underline{V}(t_o) &= \underline{V}(t_1) - \underline{V}(t_o) \\ \Delta \underline{V}(t_f) &= \underline{V}(t_f) - \underline{V}(t_2) \end{aligned} \tag{8}$$

(c) In the regions of thrust the gravitational force may be neglected.

If in addition we assume that the thrust direction is fixed the differential equations for the state variables and the Lagrange multipliers, within the burning region reduce to

$$\dot{\underline{V}} = -\frac{c\dot{m}}{m}\underline{T} \tag{9}$$

$$\ddot{\underline{\lambda}} = 0 \tag{10}$$

$$\sigma(t) = \sigma(t_A) + \int_{t_A}^{t} \dot{\sigma}\, dt \tag{11}$$

where

$$\dot{\sigma} = -\frac{c\dot{m}\lambda}{m^2} \tag{12}$$

and

$$m(t) = m(t_A) + (t - t_A)\,\dot{m} \tag{13}$$

In the burning regions the thrust vector is in the direction of $\Delta\underline{V}$. Therefore from Eq. (6) we have

$$\underline{\lambda} = \lambda \frac{\Delta\underline{V}}{\Delta V} \tag{14}$$

In the coasting region, m and σ are constant. Thus, it follows that

$$\sigma(t_1) = \sigma(t_2) \tag{15}$$

$$m(t_1) = m(t_2) \tag{16}$$

For the computations of the initial values of the Lagrange multipliers, one proceeds as follows:

First Eqs. (9) and (10) are integrated in the two burning regions t_o to t_1 and t_2 to t_f, resulting in

$$m(t_1) = m(t_o)\, e^{-\frac{\Delta V_o}{c}} \tag{17}$$

$$m(t_f) = m(t_o)\, e^{-\frac{(\Delta V_o + \Delta V_f)}{c}} \tag{18}$$

$$\dot{\underline{\lambda}}(t_1) = \dot{\underline{\lambda}}(t_o) = \text{constant} \tag{19}$$

$$\dot{\underline{\lambda}}(t_2) = \dot{\underline{\lambda}}(t_f) = \text{constant} \tag{20}$$

$$\underline{\lambda}(t_1) = \underline{\lambda}(t_o) + (t_1 - t_o)\,\dot{\underline{\lambda}}(t_o) \tag{21}$$

$$\underline{\lambda}(t_f) = \underline{\lambda}(t_2) + (t_f - t_2)\,\dot{\underline{\lambda}}(t_2) \tag{22}$$

where the time spent in the two burning regions is computed by using Eqs. (13), (16), (17), and (18), and is given by

$$(t_1 - t_o) = \frac{m(t_o)\left(e^{-\frac{\Delta V_o}{c}} - 1\right)}{\dot{m}} \tag{23}$$

$$(t_f - t_2) = \frac{m(t_o)\, e^{-\frac{\Delta V_o}{c}} \left(e^{-\frac{\Delta V_f}{c}} - 1\right)}{\dot{m}} \tag{24}$$

From the assumption that the thrust direction is fixed during each burning interval it is evident that $\underline{\lambda}$ and $\dot{\underline{\lambda}}$ are in the same direction. Therefore only the magnitude of $\underline{\lambda}$ and $\dot{\underline{\lambda}}$ need be considered, i.e. λ and $\dot{\lambda}$.

At the transition times t_1 and t_2 the switching function must be zero. Thus,

$$\frac{\lambda(t_1)}{m(t_1)} = \frac{\sigma(t_1)}{c} \tag{25}$$

and

$$\frac{\lambda(t_2)}{m(t_2)} = \frac{\sigma(t_2)}{c} \tag{26}$$

It can be shown that by integrating Eq. (11) in the two burning regions and making use of Eqs. (12) through (26) one forms the following three independent equations with five unknowns, i.e., $\sigma(t_o)$, $\lambda(t_o)$, $\dot{\lambda}(t_o)$, $\lambda(t_f)$ and $\dot{\lambda}(t_f)$

$$\frac{c}{m(t_o)} - \sigma(t_o) - \frac{\Delta V_o}{\dot{m}}\, \dot{\lambda}(t_o) = 0 \tag{27}$$

$$\frac{c e^{\frac{\Delta V_o}{c}}}{m(t_o)}\, \lambda(t_f) + \frac{c}{\dot{m}}\left(1 - e^{-\frac{\Delta V_f}{c}}\right) \dot{\lambda}(t_f) + \frac{c}{m(t_o)}\, e^{\frac{\Delta V_o}{c}} \left(e^{\frac{\Delta V_f}{c}} - 1\right) \lambda(t_o)$$

$$+ \frac{\dot{\lambda}(t_o)}{\dot{m}} \left[c\left(1 - e^{\frac{\Delta V_f}{c}}\right) + \Delta V_f\right] = 1 \tag{28}$$

$$\frac{e^{\frac{\Delta V_o}{c}}}{m(t_o)}\left[\lambda(t_o)-\lambda(t_f)\right]+\frac{1}{\dot{m}}\left(1-e^{\frac{\Delta V_o}{c}}\right)-\frac{1}{\dot{m}}\left(1-e^{-\frac{\Delta V_f}{c}}\right)\lambda^{\cdot}(t_f)=0 \qquad (29)$$

By making use of the transversality condition $\underline{\lambda}\cdot\dot{\underline{R}}-\underline{\lambda}^{\cdot}\cdot\dot{\underline{V}}+\sigma\dot{m}=0$ at times t_o and t_f one can obtain two more equations.

$$-\lambda^{\cdot}(t_o)\frac{\underline{V}(t_o)\cdot\Delta\underline{V}_o}{\Delta V_o}-\frac{c\dot{m}}{m(t_o)}\lambda(t_o)+\sigma(t_o)\,\dot{m}=0 \qquad (30)$$

$$-\lambda^{\cdot}(t_f)\frac{\underline{V}(t_f)\cdot\Delta\underline{V}_f}{\Delta V_f}-\frac{c\dot{m}}{m(t_o)}e^{\frac{\Delta V_o+\Delta V_f}{c}}\lambda(t_f)+\dot{m}=0 \qquad (31)$$

Eqs. (27) through (31) constitute five equations with five unknowns. The solution of this system of equations is given by

$$\underline{\lambda}(t_o)=\frac{m(t_o)}{c}e^{-\frac{(\Delta V_o+\Delta V_f)}{c}}\frac{\Delta\underline{V}_o}{\Delta V_o} \qquad (32)$$

$$\underline{\lambda}^{\cdot}(t_o)=0 \qquad (33)$$

$$\sigma(t_o)=e^{-\frac{(\Delta V_o+\Delta V_f)}{c}} \qquad (34)$$

$$\underline{\lambda}(t_f)=\frac{m(t_o)}{c}e^{-\frac{(\Delta V_o+\Delta V_f)}{c}}\frac{\Delta\underline{V}_f}{\Delta V_f} \qquad (35)$$

$$\underline{\lambda}^{\cdot}(t_f)=0 \qquad (36)$$

It is of interest to note that the magnitudes of $\underline{\lambda}$ at the initial and final times are equal and directly proportional to the mass at the final time. In addition, the value of σ is also proportional to the final mass and may be expressed as

$$\sigma(t)=\frac{m(t_f)}{m(t)} \qquad (37)$$

Second Method

An approach for obtaining a better first approximation is to remove or at least "relax" some of the assumptions made in the first method. More specifically, instead of completely neglecting the gravitational force in the regions of thrust it can be assumed that the gravitational force has a constant value of $\frac{-\mu \underline{R}_o}{r_o^3}$ in the first region and $\frac{-\mu \underline{R}_f}{r_f^3}$ in the second region.

In addition, we assume that the direction of the total acceleration in the two regions of thrust is parallel to the vector $\Delta \underline{V}_o$ and $\Delta \underline{V}_f$, respectively. This implies that the direction of the thrust is not fixed.

It is clear that in the region of thrust the vector $\underline{\lambda}$ lies in the plane formed by the vectors $\underline{R}$ and $\Delta \underline{V}$. It is most convenient to resolve $\underline{\lambda}$ into components along the vector $\underline{R}$ and normal to it. These two components are designated as λ_ξ and λ_η, respectively.

The differential equation for $\underline{\lambda}$ can now be written as

$$\ddot{\lambda}_\xi = \frac{2\mu}{\gamma^3} \lambda_\xi \tag{38}$$

$$\ddot{\lambda}_\eta = \frac{\mu}{r^3} \lambda_\eta \tag{39}$$

The solution to Eqs. (38) and (39) is given by

$$\lambda_\xi = \lambda_\xi(t_o) \cosh \sqrt{\frac{2\mu}{r^3}}\, t + \sqrt{\frac{r^3}{2\mu}}\, \dot{\lambda}_\xi(t_o) \sinh \sqrt{\frac{2\mu}{r^3}}\, t \tag{40}$$

$$\lambda_\eta = \lambda_\eta(t_o) \cos \sqrt{\frac{\mu}{r^3}}\, t + \sqrt{\frac{r^3}{\mu}}\, \dot{\lambda}_\eta(t_o) \sin \sqrt{\frac{\mu}{r^3}}\, t \tag{41}$$

Since the intervals of thrust are assumed to be of short duration it is permissible to approximate Eqs. (40) and (41) in the regions of thrust by neglecting the second order terms of a Taylor series expansion, i.e.,

$$\underline{\lambda}(t) \approx \underline{\lambda}(t_o) + (t - t_o)\, \dot{\lambda}(t_o) \qquad t_o \le t \le t_1 \tag{42}$$

$$\underline{\lambda}(t) \approx \underline{\lambda}(t_2) + (t - t_2)\, \dot{\lambda}(t_2) \qquad t_2 \le t \le t_f \tag{43}$$

Similarly, one can approximate $\dot{\lambda}$ in the regions of thrust to the same order of accuracy.

$$\dot{\lambda}_\xi(t) \approx \frac{2\mu}{r^3}(t - t_o)\lambda_\xi(t_o) + \dot{\lambda}_\xi(t_o) \tag{44}$$

$$t_o \le t \le t_1$$

$$\dot{\lambda}_\eta(t) \approx -\frac{\mu}{r^3}(t - t_o)\,\lambda_\eta(t_o) + \dot{\lambda}_\eta(t_o) \tag{45}$$

$$\dot{\lambda}_\xi(t) \approx \frac{2\mu}{r^3}(t - t_2)\lambda_\xi(t_2) + \dot{\lambda}_\xi(t_2) \tag{46}$$

$$t_2 \le t \le t_f$$

$$\dot{\lambda}_\eta(t) \approx -\frac{\mu}{r^3}(t - t_2)\lambda_\eta(t_2) + \dot{\lambda}_\eta(t_2) \tag{47}$$

The procedure for obtaining the initial values of the Lagrange multipliers is now the same as in the first method except that Eqs. (19) through (22) are now replaced by Eqs. (42) through (47).

CONCLUSION

A set of approximate initial values of the Lagrange multipliers have been derived. In addition, a method for obtaining a better first approximation has been outlined. It should be pointed out, however, that as one attempts to obtain these improved first approximations in the manner outlined, the algebraic manipulation of the expressions involved become more cumbersome and additional approximations may be needed.

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(ARD-896-451)
19 June 1962

First Semiannual Report

THE HAMILTON-JACOBI FORMULATION OF THE RESTRICTED THREE BODY PROBLEM IN TERMS OF THE TWO FIXED CENTER PROBLEM

RAC 720-2

Research Regarding
Guidance and Space Flight Theory
Relative to the Rendezvous Problem
Contract No. NASS-2605

REPUBLIC AVIATION CORPORATION
Farmingdale, L. I., New York

FOREWORD

This document is the First Semiannual Report prepared by Republic Aviation Corporation under NASA Contract No. NAS8-2605, "Research Regarding Guidance and Space Flight Theory Relative to the Rendezvous Problem." The contract was initiated and is monitored by W. Miner and R. Hoelker of the Aeroballistics Laboratory, George C. Marshall Space Flight Center.

The document will appear in slightly different format as a part of PROGRESS REPORT NO. 2 ON STUDIES IN THE FIELDS OF SPACE FLIGHT AND GUIDANCE THEORY, sponsored by the Aeroballistics Division of the Marshall Space Flight Center.

The report was prepared by Dr. Mary Payne and Mr. Samuel Pines of Republic's Applied Mathematics Section, Applied Research and Development Center. The authors wish to express their appreciation for many helpful discussions with Mr. Elie Lowy and Dr. George Nomicos and they especially want to thank Dr. John Morrison whose comments, from the inception of the problem, have been most illuminating.

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NOTATION

$\underline{R}$	Position vector of the vehicle relative to the barycenter in a coordinate system fixed in space
$\underline{R}_1$	Position vector of the vehicle relative to the earth
$\underline{R}_2$	Position vector of the vehicle relative to the moon
R'	Position vector of the vehicle relative to the barycenter in a rotating system
$\underline{\bar{R}}$	Position vector of the vehicle relative to the midpoint of the earth-moon line
r_1	Magnitude of $\underline{R}_1$
r_2	Magnitude of $\underline{R}_2$
Ω	Angular velocity vector of earth-moon system
ω	Magnitude of ω
μ	Gravitational constant times mass of the earth
μ'	Gravitational constant times mass of the moon
$\mathcal{L}$	Lagrangian function
$\underline{P}$	Momentum vector
H	Hamiltonian function
q_i	Generalized coordinates conjugate to p_i
Q_i	Generalized coordinates conjugate to P_i
p_i	Generalized momenta conjugate to q_i
P_i	Generalized momenta conjugate to Q_i

S_1	Time-dependent generating function	
t	Time	
H_1	Integrable part of the Hamiltonian	
H_2	Disturbing function	
h	Energy constant for H_1	
W	Time-independent generating function	
J_i	Action variables	
w_i	Angle variables	
ν_i	Frequencies for two fixed center problem	
q_1, q_2	Elliptic coordinates	prolate spheroidal coordinates
φ	Angle measured around x-axis	prolate spheroidal coordinates
c	Half the distance between earth and moon	
P_1, P_2, P_φ	Momenta conjugate to prolate spheroidal coordinates	
x, y, z	Rectangular coordinates in a system with earth at (c, o, o), moon at (-c, o, o) and $\underline{\Omega}$ in the z direction	
α	Angular momentum about the line of centers in the two fixed center problem	
β	Third dynamical constant of motion of the two fixed center problem	
$R^2(q_1)$	Fundamental quartic associated with q_1	
$S^2(q_2)$	Fundamental quartic associated with q_2	
u	Parameter in terms of which coordinates and time of the two fixed center problem are given	

r_i	Roots of $R^2(q_1) = 0$
s_i	Roots of $S^2(q_2) = 0$
n_i	Coefficient of linear term in q_i contribution to time as a function of u
m_i	Coefficient of linear term in q_i contribution to φ as a function of u
$F_i(u)$	Periodic term in time due to q_i
$G_i(u)$	Periodic term in φ due to q_i
K_1	Quarter period of q_1 elliptic functions
K_2	Quarter period of q_2 elliptic functions
Q_h	Coordinate conjugate to h
Q_α	Coordinate conjugate to α
Q_β	Coordinate conjugate to β

FINAL REPORT

CONTRACT NO. NAS 8-2605

A. CELESTIAL MECHANICS

In this study perturbation techniques using the classical Hamilton-Jacobi theory have been developed for application to the restricted three body problem. The unperturbed trajectory is based on the solution of the two fixed center problem as outlined in the First Semi-annual report RAC 720-2, submitted in June 1962.

Various approximation methods have been developed for which the initial position and velocity of a vehicle are known. The derivations have been carried out in a coordinate system rotating about an accelerated origin, which is selected so as to minimize the effects of the non-integrable terms in the perturbation equations. This procedure is presented in the Second Semi-annual report RAC 720-3, submitted in December 1962.

The various approximation methods differ in the manner in which the initial conditions are modified.

The position and velocity at time t_o are computed relative to the earth in the fixed system for two of these methods as follows:

1) Position and velocity at time t_o in the fixed system are unmodified.
2) Position $\underline{R}_{1o}'$ and velocity $\dot{\underline{R}}_{1o}'$ at time t_o in the fixed system are computed from the unmodified conditions $\underline{R}_{1o}$ and $\dot{\underline{R}}_{1o}$ as as follows:

$$\underline{R}_{1o}' = \underline{R}_{1o} + \frac{\mu'}{\mu+\mu'}\left[\underline{L}(t) - \underline{L}(o)\right] + \underline{A}(t) - \underline{A}(o)$$

$$\dot{\underline{R}}_{1o}{}' = \dot{\underline{R}}_{1o} - \frac{\mu'}{\mu + \mu'} \underline{L}_M (o) + \underline{\Omega} \times \underline{A}(t)$$

in which $\underline{L}$ is the vector from earth to moon, $\dot{\underline{L}}$ is the velocity vector of the moon relative to the earth and the vector $\underline{A}$ is given by the formula

$$\underline{A} = \alpha \, \underline{L}(t) + \gamma \, \dot{\underline{L}}(t)$$

and is determined from initial and final positions associated with t_o and t, respectively, so as to minimize, in a least square sense, the initial and final contribution of the non-integrable parts of the perturbation equations. In the above methods the earth and moon are considered fixed in their final positions.

A second perturbation theory has also been formulated in an inertial coordinate system, where the fixed positions for the earth and moon have been selected so as to reduce the effect of the non-integrable terms in the perturbation equations.

Small variations in the parameters α, γ, δ, and ϵ lead to modifications of initial conditions which would improve the approximation of the restricted problem by the two fixed center problem. This theory is contained in the Third Semiannual Report, RAC No. 720-5, submitted in August 1963.

The application of the qualitative theory of differential equations to problems in celestial mechanics has been explored and methods for analyzing periodic solutions of differential equations have been investigated with a view towards obtaining qualitative information about the motions in a gravitational field of several bodies.

The application of the variation of parameters to the polar oblateness problem has been investigated. The following set of two body parameters and their associated perturbation equations have been derived:

1) The perigee vector $\underline{P}(t)$ of the instantaneous Kepler orbit
2) The tangent vector $\underline{Q}(t)$ to this orbit at perigee
3) The time of perigee passage $\sigma(t)$

A scheme has been devised to evaluate the first order perturbations on the motion of a space vehicle caused by polar oblateness of the earth. The parameters $\underline{P}(t)$, $\underline{Q}(t)$ and $\sigma(t)$ have been expressed in terms of the state variables, $\underline{R}$ and $\dot{\underline{R}}$ of Kepler motion and the perturbations to the Kepler orbit resulting from the application of a general force $\underline{F}$ have been derived. The general perturbation equations have been used to develop a method for evaluating the orbital perturbations of a space vehicle due to the polar oblateness of the earth. It has been found that the terms of the equation with long periods can be written in closed form.

A modified Poisson method has been used to obtain the first order solution to the problem. The modification of the method is introduced in order to eliminate the occurrence of secular terms which cause a rapid deterioration of the solution.

The approximate solution is expressed as a function of true anomaly and some analysis of second order theory suggests that difficulties with particular initial conditions may be avoided. The details of this derivation are given in the Fourth Semi-annual Report, RAC 720-7, submitted in February 1964.

B. CALCULUS OF VARIATIONS

In this study a differential correction scheme has been developed for the improvement of the approximate initial values of the adjoint variables (Lagrange multipliers) so that an integral functional satisfying the desired boundary conditions is optimized. The adjoint variables satisfy a system of equations that are developed by applying the classical methods of the calculus of variations, properly extended, or Pontryagin's maximum principle.

A general transition matrix has been derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program and of the final time of the nominal optimum trajectory.

An iteration scheme has also been outlined for the convergence of the differential corrections to the desired end conditions.

A method has been established for obtaining approximate initial values of the Lagrange multipliers in the "Two Point Boundary Value Problem of the

Calculus of Variations". In this method the following assumptions have been made:

1) Two burning periods are required to accomplish the optimum trajectory.

2) The time intervals in these two regions of thrust are small so that the changes in velocity can be obtained from the solution to the "Two-Impulse Orbital Transfer" problem.

3) In the regions of thrust the gravitational force may be neglected.

In order to improve this method, the last assumption has been modified and the gravitational acceleration is not neglected but is regarded as a constant vector in each of the burning regions.

The details of the derivations are incorporated in the Second, Third and Fourth Semi-annual Reports, RAC 720-4, 6, and 8 submitted in December 1962, August 1963, and February 1964, respectively.

RAC-720-6
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Part II of

Third Semiannual Report

DIFFERENTIAL CORRECTION SCHEME FOR THE CALCULUS OF VARIATIONS

RAC 720-6

Research Regarding
Guidance and Space Flight Theory
Relative to the Rendezvous Problem
Contract No. NAS 8-2605

REPUBLIC AVIATION CORPORATION
Farmingdale, L.I., N.Y.

REPUBLIC AVIATION CORPORATION
Farmingdale, L.I., N.Y.

DIFFERENTIAL CORRECTION SCHEME FOR THE CALCULUS OF VARIATIONS

by

George N. Nomicos

SUMMARY

A differential correction scheme is developed for the improvement of the approximate initial values of the adjoint variables so that an integral functional satisfying desired boundary conditions is optimized. The adjoint variables satisfy a system of equations that are developed by applying the classical methods of the calculus of variations, properly extended, or Pontryagin's maximum principle. Approximate initial values for the adjoint variables are assumed.

A general transition matrix is derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program and of the final time of the nominal optimum trajectory. An iteration scheme also is discussed for the convergence of the differential corrections to the desired end conditions.

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SUMMARY

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LIST OF SYMBOLS

a	Semimajor axis of Kepler orbit
c	Gas exhaust velocity
E	Eccentric anomaly
$\underline{e}$	Unit vector along the thrust direction
$F(t)$	Partials of the vector functions $\underline{f}$ and $\underline{g}$ with respect to the vectors of state variables $\underline{x}$ and adjoint variables $\underline{y}$
$f_o(\underline{x}, \underline{u})$	Integral functional to be optimized
$\underline{f}(\underline{x}, \underline{u})$	Vector function of state variables (n-dimensional)
$\underline{f}(\underline{x}, \underline{u}, \underline{y})$	General form of vector state variables
$f, g, \dot{f}, \dot{g}$	Scalar functions relating position and velocity vectors a time t with initial position and velocity vectors for the Kepler problem
$\underline{g}(\underline{x}, \underline{u}, \underline{y})$	General form of vector adjoint variables
$\mathcal{H}(\underline{x}, \underline{u}, \underline{y})$	Hamiltonian
$\underline{H}$	Angular momentum vector $\underline{R} \times \dot{\underline{R}}$
h	Magnitude of angular momentum
$m(t)$	Mass of space vehicle
N	Number of switchings of thrusting program
n	Mean motion
$\underline{R}$	Position vector of the vehicle
$\dot{\underline{R}}$	Velocity vector of the vehicle
$P(t)$	Transformation of variations of conventional state variables to those of the orbit parameters

$\underline{r}$	General vector of state and adjoint variables
r	Magnitude of position vector
S(t)	Switching function for engine, "on" or "off"
T	Final time
t	Time
$\underline{u}(t)$	Control function of time
U	Control region (independent of time)
$\underline{V}$	Velocity vector of vehicle
v	Magnitude of velocity vector
$X_{x,y}(T, t_o)$	Transition matrix of the partials $\frac{\partial \underline{x}(T)}{\partial \underline{x}(t_o)}$ and $\frac{\partial \underline{x}(T)}{\partial \underline{y}(t_o)}$
$x_o(T)$	Integral to be optimized
$\underline{x}(t)$	State vector variables (n-dimensional)
$\underset{\sim}{x}(t)$	Augmented state vector $(x_o, \underline{x})$
$\underline{y}(t)$	Vector of adjoint variables (n-dimensional)
$\underset{\sim}{y}(t)$	Augmented vector of adjoint variables $(y_o, \underline{y})$
$Y_{x,y}(T, t_o)$	Transition matrix of the partials $\frac{\partial \underline{y}(T)}{\partial \underline{x}(t_o)}$ and $\frac{\partial \underline{y}(T)}{\partial \underline{y}(t_o)}$ respectivly

GREEK LETTERS

$\underline{\alpha}(t)$	Set of orbit parameters
$[\Gamma(T, t_o)]$	General transition matrix of $\frac{\partial \underline{x}(T)}{\partial \underline{y}(t_o)}$ including the optimum change of thrusting program
$[\hat{\Gamma}]$	The first six rows of the general transition matrix $[\Gamma]$
Γ_7	The last row of the general transition matrix $[\Gamma]$
$\delta\dot{\underline{x}}(t_j)$	$\lim_{\epsilon \to o} [\dot{\underline{x}}(t_j - \epsilon) - \dot{\underline{x}}(t_j + \epsilon)]$ at time t_j of change of thrusting program
$\delta\dot{\underline{y}}(t_j)$	$\lim_{\epsilon \to o} [\dot{\underline{y}}(t_j - \epsilon) - \dot{y}(t_j + \epsilon)]$ at time t_j of change of thrusting program
δ_{ij}	Kroneker's delta
$\Delta\underline{\alpha}(t)$	Variation of the set of orbit parameters

$\Delta\underline{h}(t)$	Variation of the general vector of state and adjoint variables due to the control vector change $\Delta\underline{u}$
$\Delta\underline{f}(t)$	Variation of the vector function of the state variables due to control vector change $\Delta\underline{u}$
$\Delta\underline{g}(t)$	Variation of the vector function of the adjoint variables due to the control vector change $\Delta\underline{u}$
$\Delta\underline{r}(t)$	Variation of the general vector of state and adjoint variables
$\Delta S(t)$	Variation of the switching function $S(t)$
ΔT	Variation of the final time T
θ	Eccentric anomaly measured from initial position
$\underline{\lambda}$	Vector of adjoint variables (y_4, y_5, y_6)
μ	Gravitational constant times mass of the attracting body
$\underline{\nu}$	Vector of adjoint variables (y_1, y_2, y_3)
$\Phi(t, t_o)$	Transition matrix relating variations of the state variables $\underline{x}$ and the adjoint variables $\underline{y}$ at time t with those at t_o
$\Psi(t, t_o)$	Transition matrix of the set of orbit parameters
$[\Omega]$	General transition matrix of $\frac{\partial \underline{y}(T)}{\partial \underline{y}(t_o)}$ including the optimum change of the thrusting program
$[\hat{\Omega}]$	The first six rows of the general transition matrix $[\Omega]$
Ω_7	The last row of the general transition matrix $[\Omega]$

SUBSCRIPTS

i, j	Components
o	Initial value of time t_o

SUPERSCRIPTS

•	Differentiation wrt time
T	Transpose of a matrix
^	Vector or matrix reduced to six rows
-1	Inverse of a matrix

INTRODUCTION

In the problems of the calculus of variations, a system of partial differential equations must be solved with specified boundary conditions. In addition to the state and control variables that appear in the equations of motion, the inequalities of constraints, and the functional that should be optimized, there is a number of adjoint variables that satisfy additional equations for the optimization of the given system. These equations are derived by the application of the classical methods of the calculus of variations, properly extended, or from Pontryagin's maximum principle [1], [2].

When some approximate values of the adjoint variables at the initial time t_o have been calculated, then, by numerical integration of the above systems of equations, an optimal solution is obtained that does not satisfy the desired end conditions. In this paper, a differential correction scheme is developed that will improve the approximate initial values of the adjoint variables so that the optimal solution will satisfy the desired end conditions. A general transition matrix is derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program of the nominal optimum trajectory and the variation of the final time. An iteration scheme also is presented for the convergence of the improved values of the adjoint variables to those of the optimum solution.

First, the general equations of the state variables, used mostly as constraints, are given, together with the equations of the adjoint variables. Second, the variational equations for the above systems of equations are derived, and an application to the problem of minimizing the fuel of a space vehicle flying between two given boundary points is given as an example. Third, a differential correction scheme is derived for the improvement of the approximate initial values of the adjoint variables, and an iteration scheme is presented for the convergence of the improved values of the adjoint variables, so that the optimum solution will satisfy the desired end conditions. Finally, conclusions and recommendations are presented for the application of this scheme to the actual flight of space vehicles.

FUNDAMENTAL SYSTEM OF EQUATIONS

State Variables

The motion of a vehicle is characterized by the vector variable $\underline{x}(t)$ belonging to the vector space W at any instant of time t. It is assumed that this motion is controlled by a control vector $\underline{u}(t)$.

The fundamental system of equations of state variables is given by

$$\dot{x}_i(t) = f_i(\underline{x}(t), \underline{u}(t)) \qquad (i = 1, 2, \ldots n) \tag{1}$$

where $\underline{x}(t)$ is an n-dimensional piecewise differentiable state vector, and $\underline{u}(t)$ is an r-dimensional piecewise continuous control vector belonging to an arbitrary control region U that is independent of time. The functions f_i are defined for $\underline{x} \in W$ and for $\underline{u} \in U$ and are assumed to be continuous in the variables $\underline{x}(t)$ and $\underline{u}(t)$ and continuously differentiable with respect to $\underline{x}(t)$. For a certain admissible control $\underline{u}(t)$, the motion of the vehicle $\underline{x}(t)$ is uniquely determined.

The integral functional to be optimized is

$$x_o(T) = \int_{t_o}^{T} f_o(\underline{x}(t), \underline{u}(t))dt \tag{2}$$

The necessary conditions for the optimum control vector $\underline{u}(t)$ of Eq.(2) are formulated for fixed boundary conditions of the state variables $\underline{x}(t_o)$ and $\underline{x}(T)$ and for free end time T.

Adjoint Variables

For the optimum solution of Eq. (2), another system of equations is considered. This system is linear and homogeneous in the adjoint variables $\underline{y}(t) = (y_o, y_1, \ldots y_n) = (y_o, \underline{y})$ which is an (n+1)-dimensional continuous vector, and is given by

$$\dot{y}_i(t) = -\sum_{j=0}^{n} \frac{\partial f_j(\underline{x}(t), \underline{u}(t))}{\partial x_i} y_j(t) \qquad (i = 0, 1, \ldots n) \tag{3}$$

The Hamiltonian $\mathcal{H}(\underset{\sim}{x}(t), \underline{u}(t), \underset{\sim}{y}(t))$ is defined by

$$\mathcal{H}(\underset{\sim}{x}, \underline{u}, \underset{\sim}{y}) = \sum_{i=0}^{n} y_i(t) f_i(\underline{x}(t), \underline{u}(t)) \tag{4}$$

and the systems of Eqs. (1), (2), and (3) correspond to the Hamiltonian system

$$\begin{aligned} \dot{x}_i(t) &= \frac{\partial \mathcal{H}}{\partial y_i} \\ \dot{y}_i(t) &= \frac{-\partial \mathcal{H}}{\partial x_i} \end{aligned} \tag{5}$$

Pontryagin's maximum principle and transversality condition give, for optimal $x_o(T)$, the function $\mathcal{H}(\underset{\sim}{x}(t), \underline{u}(t), \underset{\sim}{y}(t))$ of $\underline{u}(t)$ belonging to U attains its maximum at the point $\underline{u}(t)$, i.e.

$$\begin{gathered} \mathcal{H}(\underset{\sim}{x}(t), \underline{u}(t), \underset{\sim}{y}(t)) = \sup_{\underline{u} \in U} \mathcal{H}(\underset{\sim}{x}(t), \underline{u}(t), \underset{\sim}{y}(t)) = 0 \\ y_o(t_o) \leq 0 \quad \text{and} \quad y_k(T) = 0 \end{gathered} \tag{6}$$

where the subscript k corresponds to the subscript of the state variables for which the terminal value $x_k(T)$ is free. For most of the engineering applications, we have $y_o \neq 0$, which is normalized to $y_o = -1$.

The Lagrangian multipliers $\overset{(L)}{\underline{\lambda}}(t)$ of the classical calculus of variations are related to the adjoint variables $\underline{y}(t)$ by the relationship

$$\underset{i \neq 0}{\overset{(L)}{\lambda_i}}(t) = \frac{\partial f_o(\underline{x}(t), \underline{\dot{x}}(t), \underline{u}(t))}{\partial \dot{x}_i} y_o(t) + y_i(t) \tag{7}$$

If the time t appears explicitly in the system of functions f or f_o, then it always can be transformed to an autonomous system by introducing an auxiliary state variable that is defined by

$$\dot{x}_{n+1}(t_o) = 1 \qquad \text{with} \qquad x_{n+1}(t_o) = t_o \tag{8}$$

Example

For a space vehicle powered by a throttled engine and flying in the gravitational field of only one attracting body, the system of equations of the state variables, i.e., Eq. (1), reduces to

$$
\begin{aligned}
&\dot{\underline{R}} = \underline{V} && f_1, f_2, f_3 \\
&\dot{\underline{V}} = -\frac{\mu}{r^3}\underline{R} + \frac{u(t)}{m}\underline{e} && f_4, f_5, f_6 \\
&\dot{m} = -\frac{u(t)}{c} && f_7
\end{aligned}
\tag{9}
$$

where $\underline{e}$ is a unit vector in the direction of the thrust, and $u(t)$ is the control variable belonging to the range $0 \leq u(t) \leq K$.

For minimizing the fuel between $\underline{x}(t_o)$ and $\underline{x}(T)$ with free end time, the integral functional to be optimized, i.e., Eq. (2), becomes

$$
x_o(T) = \int_{t_o}^{T} f_o(\underline{x}(t), u(t))dt \tag{10}
$$

with $f_o = -\dot{m} = \frac{u(t)}{c}$.

The system of the adjoint variables, i.e., Eq. (3), reduces to

$$
\begin{aligned}
&\dot{y}_o(t) = 0 && \\
&\dot{\underline{\nu}}(t) = \frac{\mu}{r^3}\underline{\lambda} - 3\mu\frac{\underline{R}\cdot\underline{\lambda}}{r^5}\underline{R} && \underline{\nu} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&\dot{\underline{\lambda}}(t) = -\underline{\nu} && \\
&\dot{y}_7(t) = \frac{u(t)}{m^2}(\underline{\lambda}\cdot\underline{e}) && \underline{\lambda} = \begin{bmatrix} y_4 \\ y_5 \\ y_6 \end{bmatrix}
\end{aligned}
\tag{11}
$$

The maximum principle and the transversality conditions of Eq. (6) become

$$\mathcal{H} = \sup_{u \in U} \mathcal{H} = y_o f_o + \underline{\nu} \cdot \underline{V} + \underline{\lambda} \cdot \left(\frac{\mu}{r^3} \underline{R} + \frac{u(t)}{m} \underline{e} \right) - y_7 \frac{u(t)}{c} = 0$$

$$y_o(t) = -1 \qquad \text{and} \qquad y_7(T) = 0 \tag{12}$$

where $f_o = \frac{u(t)}{c}$.

From Eq. (1), it is obvious that $\underline{\lambda} // \underline{e}$ and that the switching function for $u = 0$ or $u = K$ is defined by

$$S(t) = \frac{|\lambda|}{m} - \frac{y_7 - y_o}{c} \gtrless 0 \tag{13}$$

when $u(t) = \begin{cases} K \ (\max) \\ 0 \ (\min) \end{cases}$ respectively.

VARIATIONAL EQUATIONS

In this section, the variational equations of the optimum trajectory of a space vehicle are derived. The formulation of these equations is required for the application of the differential correction scheme that is developed in the next section.

The application of Pontryagin's maximum principle for the solution of optimal problems yields additional information for the synthesis of optimal controls. Making use of this principle, the system of Eqs. (1) and (3) may be rewritten in the following general form.

$$\dot{\underline{r}}(t) \equiv \begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\underline{y}}(t) \end{bmatrix} = \begin{bmatrix} \underline{f}(\underline{x}, \underline{y}, \underline{u}) \\ \underline{g}(\underline{x}, \underline{y}, \underline{u}) \end{bmatrix} \tag{14}$$

The variations of this system are obtained by

$$\Delta \dot{\underline{r}}(t) = F(t)\, \Delta \underline{r}(t) + \Delta \underline{h}(t) \tag{15}$$

where the matrix $F(t)$ and the vector $\Delta \underline{h}(t)$ are given by

$$F(t) = \begin{bmatrix} \dfrac{\partial \underline{f}}{\partial \underline{x}} & \dfrac{\partial \underline{f}}{\partial \underline{y}} \\[2ex] \dfrac{\partial \underline{g}}{\partial \underline{x}} & \dfrac{\partial \underline{g}}{\partial \underline{y}} \end{bmatrix}$$

$$\Delta \underline{h}(t) = \begin{bmatrix} \Delta \underline{f} \\[2ex] \Delta \underline{g} \end{bmatrix} = \begin{bmatrix} \underline{f}(\underline{u} + \Delta \underline{u}) - \underline{f}(\underline{u}) \\[2ex] \underline{g}(\underline{u} + \Delta \underline{u}) - \underline{g}(\underline{u}) \end{bmatrix} . \tag{16}$$

Transition Matrix

The fundamental solution matrix for the homogeneous part of Eq. (15), i.e.,

$$\dot{\Phi}(t) = F(t)\, \Phi(t)$$

with initial conditions $\Phi(t_o, t_o) = I$ (unit matrix), is the transition matrix $\Phi(t, t_o)$ of the system. From the properties of the fundamental solution matrix and the transition matrix $\Phi(t, t_o)$, we obtain

$$\Delta \underline{r}(t) = \Phi(t, t_o)\, \Delta \underline{r}(t_o) + \int_{t_o}^{t} \Phi(t, \tau)\, \Delta \underline{h}(\tau)\, d\tau \tag{17}$$

which is the solution of the non-homogeneous Eq. (15).

In the example of the powered space vehicle flying in the gravitational field of one attracting body, Eq. (17) reduces to

$$\Delta \underline{r}(T) = \Phi(T, t_o)\, \Delta \underline{r}(t_o) + \sum_{j=1}^{N} \Phi(T, t_j)\, \Delta \underline{h}(t_j)\, \Delta t_j \tag{18}$$

where t_j is the time at which the thrusting program of the optimum nominal trajectory with the approximate values of initial conditions $\underline{r}(t_o)$ switches "on" or "off" during the time interval $t_o < t_j < T$, and $\Delta \underline{r}(T)$ gives the deviations of the nominal end conditions from the desired end conditions, i.e.

$$\Delta \underline{r}(T) = \begin{bmatrix} \Delta \underline{x}(T) \\ \Delta \underline{y}(T) \end{bmatrix}$$

$$\Phi(T, t_o) = \begin{bmatrix} \dfrac{\partial \underline{x}(T)}{\partial \underline{x}(t_o)} & \dfrac{\partial \underline{x}(T)}{\partial \underline{y}(t_o)} \\ \dfrac{\partial \underline{y}(T)}{\partial \underline{x}(t_o)} & \dfrac{\partial \underline{y}(T)}{\partial \underline{y}(t_o)} \end{bmatrix} = \begin{bmatrix} X_x(T, t_o) & X_y(T, t_o) \\ Y_x(T, t_o) & Y_y(T, t_o) \end{bmatrix} \tag{19}$$

$$\Delta \underline{h}(t_j) = \lim_{\epsilon \to 0} \begin{bmatrix} \dot{\underline{x}}(t_j - \epsilon) - \dot{\underline{x}}(t_j + \epsilon) \\ \dot{\underline{y}}(t_j - \epsilon) - \dot{\underline{y}}(t_j + \epsilon) \end{bmatrix} = - \begin{bmatrix} \delta \dot{\underline{x}}(t_j) \\ \delta \dot{\underline{y}}(t_j) \end{bmatrix}$$

Because the boundary conditions of the state variables at the initial time t_o are given, we have $\Delta \underline{x}(t_o) \equiv 0$, and Eq. (18) becomes (see Fig. 1)

$$\Delta \underline{r}(T) = \Phi(T, t_o) \Delta \underline{r}(t_o) - \sum_{j=1}^{N} \Phi(T, t_j) \, \delta \dot{\underline{r}}(t_j) \, \Delta t_j \tag{20}$$

or

$$\begin{bmatrix} \Delta \underline{x}(T) \\ \Delta \underline{y}(T) \end{bmatrix} = \begin{bmatrix} X_x & X_y \\ Y_x & Y_y \end{bmatrix} \begin{bmatrix} 0 \\ \Delta \underline{y}(t_o) \end{bmatrix} - \sum_{j=1}^{N} \begin{bmatrix} X_x^{(j)} & X_y^{(j)} \\ Y_x^{(j)} & Y_y^{(j)} \end{bmatrix} \begin{bmatrix} \delta \dot{\underline{x}}(t_j) \\ \delta \dot{\underline{y}}(t_j) \end{bmatrix} \Delta t_j \tag{21}$$

where $X = X(T, t_o)$, and $X^{(j)} = X(T, t_j)$.

From Eq. (21), we get

$$\Delta \underline{x}(T) = X_y \Delta \underline{y}(t_o) - \sum_{j=1}^{N} \left[X_x^{(j)} \delta \dot{\underline{x}}(t_j) + X_y^{(j)} \delta \dot{\underline{y}}(t_j) \right] \Delta t_j \tag{22}$$

and

$$\Delta \underline{y}(T) = Y_y \Delta \underline{y}(t_o) - \sum_{j=1}^{N} \left[Y_x^{(j)} \delta \dot{\underline{x}}(t_j) + Y_y^{(j)} \delta \dot{\underline{y}}(t_j) \right] \Delta t_j \tag{23}$$

Thrusting Program

In the formulation of the variational equations of the optimum nominal trajectory, the time variation Δt_j of the optimum thrusting program has been included where t_j is the time at which the thrust switches "on" or "off" and the switching function of the nominal trajectory is zero, i.e., $S(t_j) = 0$. The time variation Δt_j is calculated from the variation of the switching function $\Delta S(t_j + \Delta t_j)$ for which

$$S(t_j + \Delta t_j) + \Delta S(t_j + \Delta t_j) = 0 \tag{24}$$

From the linear expansion of Eq. (24) we get

$$\dot{S}(t_j) \Delta t_j \simeq - \frac{\partial S}{\partial \underline{r}} \Delta \underline{r}(t_j + \Delta t_j) \tag{25}$$

Because $\Delta \underline{r}(t_j + \Delta t_j) \simeq \Delta \underline{r}(t_j) + \Delta \dot{\underline{r}}(t_j) \Delta t_j$ and $\frac{\partial S}{\partial \underline{r}} \Delta \dot{\underline{r}}(t_j) = 0$, Eq. (25) becomes

$$\dot{S}(t_j) \Delta t_j \simeq - \frac{\partial S}{\partial \underline{r}} \Delta \underline{r}(t_j) \tag{26}$$

Expanding the variation $\Delta r(t_j)$ from Eq. (20), we get

$$\Delta \underline{r}(t_j) = \Phi(t_j, t_o) \Delta \underline{r}(t_o) - \sum_{i=1}^{i<j} \Phi(t_j, t_i) \delta \dot{\underline{r}}(t_i) \Delta t_i \tag{27}$$

$$\Delta t_j = \frac{-1}{\dot{S}(t_j)} \frac{\partial S(t_j)}{\partial \underline{r}(t_j)} \left[\Phi(t_j, t_o) \Delta \underline{r}(t_o) - \sum_{i=1}^{i<j} \Phi(t_j, t_i) \delta \underline{\dot{r}}(t_i) \Delta t_i \right] \tag{28}$$

and, in terms of the variations $\Delta \underline{y}(t_o)$, it becomes

$$\begin{aligned} \Delta t_j = & - \frac{1}{\dot{S}(t_j)} \left[\frac{\partial S(t_j)}{\partial \underline{x}(t_j)} X_y(t_j, t_o) + \frac{\partial S(t_j)}{\partial \underline{y}(t_j)} Y_y(t_j, t_o) \right] \Delta \underline{y}(t_o) \\ & + \frac{1}{\dot{S}(t_j)} \frac{\partial S(t_j)}{\partial \underline{x}(t_j)} \sum_{i=1}^{i<j} \left[X_x(t_j, t_i) \delta \underline{\dot{x}}(t_i) + X_y(t_j, t_i) \delta \underline{\dot{y}}(t_i) \right] \Delta t_i \\ & + \frac{1}{\dot{S}(t_j)} \frac{\partial S(t_j)}{\partial \underline{y}(t_j)} \sum_{i=1}^{i<j} \left[Y_x(t_j, t_i) \delta \underline{\dot{x}}(t_i) + Y_y(t_j, t_i) \delta \underline{\dot{y}}(t_i) \right] \Delta t_i \end{aligned} \tag{29}$$

From Eq. (13) for the switching function $S(t)$, we find that

$$\begin{aligned} & S(t) = \frac{|\lambda|}{m} - \frac{y_7 - y_o}{c} \qquad \dot{S}(t) = \frac{\underline{\lambda} \cdot \dot{\underline{\lambda}}}{m|\lambda|} \\ & \frac{\partial S(t_j)}{\partial \underline{x}(t_j)} = \left\{ 0, 0, 0, 0, 0, 0, -\frac{|\lambda|}{m^2} \right\} \\ & \frac{\partial S(t_j)}{\partial \underline{y}(t_j)} = \left\{ \frac{y_4}{m|\lambda|}, \frac{y_5}{m|\lambda|}, \frac{y_6}{m|\lambda|}, 0, 0, 0, -\frac{1}{c} \right\} \end{aligned} \tag{30}$$

DIFFERENTIAL CORRECTION SCHEME

Correction Scheme

In this section, a differential correction scheme is developed for the improvement of the approximate initial values of the adjoint variables so that the optimum solution of the problem can be found. The variations of the nominal optimum trajectory of the space vehicle, calculated for the approximate initial values of the adjoint variables, have been derived previously.

Making use of Eqs. (17), we solve for $\Delta \underline{r}(t_o)$ if we know the variation $\Delta \underline{r}(T)$ at the terminal time T. In the example of the powered space vehicle we derived Eqs. (22) and (23) for the variations of $\Delta \underline{x}(T)$ and $\Delta \underline{y}(T)$ caused by the variations of the adjoint variables $\Delta \underline{y}(t_o)$ at the initial time t_o and the variations Δt_j at the time t_j of the thrusting program, which corresponds to the optimum nominal trajectory for the approximate adjoint variables.

Free End Time

In the case of free end time T, a variation in the terminal time also is taken into consideration, and, making use of Eqs. (29), we find that

$$\Delta \underline{x}(T) = [\Gamma]\, \Delta \underline{y}(t_o) + \dot{\underline{x}}(T)\, \Delta T \tag{31}$$

$$\Delta \underline{y}(T) = [\Omega]\, \Delta \underline{y}(t_o) + \dot{\underline{y}}(T)\, \Delta T \tag{32}$$

Separating the seventh row of Eqs. (31) and (32), we get

$$\Delta \hat{\underline{x}}(T) = [\hat{\Gamma}]\, \Delta \underline{y}(t_o) + \dot{\hat{\underline{x}}}(T)\, \Delta T \tag{33}$$

$$\Delta y_7(T) = \Omega_7\, \Delta \underline{y}(t_o) + \dot{y}_7(T)\, \Delta T \tag{34}$$

where Eqs. (33) and (34) are of the form

$$[6 \times 1] = [6 \times 7]\,[7 \times 1] + [6 \times 1]\,[1 \times 1]$$

$$[1 \times 1] = [1 \times 7]\,[7 \times 1] + [1 \times 1]\,[1 \times 1]$$

respectively, $[\hat{\Gamma}]$ represents the first six rows of $[\Gamma]$, and Ω_7 represents the seventh row of $[\Omega]$.

For the solution of the system of Eqs. (33) and (34) for $\Delta \underline{y}(t_o)$ and ΔT from the deviations $\Delta \hat{\underline{x}}(T)$ and $\Delta y_7(T) = 0$, we need one more relationship, and this is obtained from Eq. (12), i.e.

$$\mathcal{H}(\underline{x}, \underline{u}, \underline{y}) = \sum_{j=1}^{7} y_j \cdot f_j(t) - f_o(t) = 0 \tag{35}$$

Taking the variation of $\mathcal{H}(t)$ at time t_o, we get

$$\sum_{j=1}^{7} f_j(t_o)\, \Delta y_j(t_o) + \sum_{j=1}^{7} y_j(t_o)\, \Delta f_j(t_o) - \Delta f_o(t_o) = 0 \tag{36}$$

Because $\Delta f_j(t_o) = 0$ and $\Delta f_o(t_o) = 0$ if the variation of the switching function $\Delta S(t_o)$ does not change the sign of $S(t_o)$, Eq. (36) becomes

$$\sum_{j=1}^{7} f_j(t_o)\, \Delta y_j(t_o) = 0 \tag{37}$$

or

$$\underline{V}(t_o) \cdot \Delta \underline{\nu}(t_o) + \ddot{\underline{R}}(t_o) \cdot \Delta \underline{\lambda}(t_o) - \frac{u(t_o)}{c} \Delta y_7(t_o) = 0 \tag{38}$$

Thus, combining Eqs. (33), (34), and (38), we get eight equations with eight unknown variations that are given by

$$\begin{bmatrix} \Delta \hat{\underline{x}}(T) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} [\hat{\Gamma}] & \dot{\hat{\underline{x}}}(T) \\ \Omega_7 & \dot{y}_7(T) \\ \dot{\underline{x}}(t_o)^T & 0 \end{bmatrix} \begin{bmatrix} \Delta \underline{y}(t_o) \\ \\ \Delta T \end{bmatrix} \tag{39}$$

Solving for $\Delta \underline{y}(t_o)$ and ΔT, we find that

$$\begin{bmatrix} \Delta \underline{y}(t_o) \\ \\ \Delta T \end{bmatrix} = \begin{bmatrix} [\hat{\Gamma}] & \dot{\hat{\underline{x}}}(T) \\ \Omega_7 & \dot{y}_7(T) \\ \dot{\underline{x}}(t_o)^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \Delta \hat{\underline{x}}(T) \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

Iteration Scheme

For the calculation of the optimum trajectory of a space vehicle, the differential correction scheme described in this section is applied, and the variation of the adjoint vector $\Delta \underline{y}(t_o)$ at the initial time t_o, as well as the variation of the final time ΔT, are derived to match the desired conditions at the final time T in space. Making use of the corrected adjoint variables $\underline{y}_1(t_o) = \underline{y}(t_o) + \Delta\underline{y}(t_o)$, a new optimum nominal trajectory is computed by integrating the system of equations of the state and adjoint variables, i.e., Eqs. (9) and (11), by making use of Eq. (13) for the optimum thrusting program as described previously. Because the differential correction scheme has been derived for linear variations of highly nonlinear equations, it is expected that there still will be a discrepancy between the desired and the new computed values of the end conditions $\Delta\underline{x}_1(T_1)$, where $T_1 = T + \Delta T$.

In general, successive iterations generate corrections $\Delta\underline{y}_k(t_o)$ to the adjoint variables at time t_o from $\Delta\underline{x}_k(T_k)$ such that

$$\underline{y}_{k+1}(t_o) = \underline{y}_k(t_o) + \Delta\underline{y}_k(t_o) = \underline{y}(t_o) + \sum_{i=0}^{k} \Delta\underline{y}_i(t_o) \tag{41}$$

which, in turn, gives end conditions with deviations $\Delta\underline{x}_{k+1}(T_{k+1})$ from their desired values, and

$$T_{k+1} = T + \sum_{i=0}^{k} \Delta T_i \tag{42}$$

This iteration scheme converges to the desired end conditions of the state vector, provided that the deviations are within the linear range. Departure from the linear range will be indicated when the deviations of the computed nominal end conditions from the desired end conditions $\Delta\underline{x}_1(T_1)$ are comparable to or exceed the deviations $\Delta\underline{x}(T)$. In this case, each step of the iteration scheme described above contains a sub-iteration carried out on a parameter γ_k introduced as a factor multiplying the deviations $\Delta\underline{x}_k(T_k)$. Thus

$$\Delta\underline{x}_k^*(T_k) = \gamma_k \Delta\underline{x}_k(T_k) \tag{43}$$

From $\Delta\underline{x}_k^*(T_k)$, we obtain the correction $\Delta\underline{y}_k^*(t_o)$, which is added to $\underline{y}_k^*(t_o)$ for the k^{th} estimate of the adjoint variables at time t_o. The sub-iteration consists of the determination of a value of γ_k $(0 < \gamma_k \leq 1)$ such that the deviations $\Delta\underline{x}_{k+1}(T_{k+1})$ computed from the corrected adjoint variables, i.e.

$$\underline{y}_{k+1}(t_o) = \underline{y}_k(t_o) + \Delta\underline{y}_k^*(t_o) = \underline{y}(t_o) + \sum_{i=0}^{k} \Delta\underline{y}_i^*(t_o) \tag{44}$$

are comparable to or less than the deviations $\Delta\underline{x}_k(T_k)$. This procedure is continued until the linear range is reached for which $\gamma_k = 1$ and the iteration scheme converges to the desired end conditions.

It should be noted that the same procedure is followed when parameters other than the state variables are specified as end conditions. Of course, these parameters must be expressible as functions of the state variables.

CONCLUSIONS AND RECOMMENDATIONS

A differential correction has been developed for the improvement of the approximate values of the adjoint variables so that the optimal solution of the problems of the calculus of variations is obtained. The mathematical analysis for the differential correction scheme for the optimum trajectory of a space vehicle with minimum fuel consumption between fixed boundary conditions has been presented. The method developed relies on the variations of the nominal optimum trajectory of the space vehicle calculated for the approximate initial values of the adjoint variables, which are assumed to be given. Techniques for the calculation of these approximate values are not considered in this report.

A general transition matrix has been derived for the variations of the end conditions caused by the variations of the initial values of the adjoint variables, including the variations of the thrusting program of the nominal optimum trajectory and the variation of the final time. An iteration scheme also has been discussed for the convergence of the improved values of the adjoint variables to those of the optimum problem satisfying the desired end conditions. In addition, a method for the case of variations beyond the linear range has been outlined.

This program will be highly useful for the determination of optimum space missions and for optimum orbit transfer for intercept and rendezvous of space vehicles as well as for optimum navigation and guidance of a space vehicle. Further work in this area is readily suggested. First, techniques should be developed for the approximate initial values of the adjoint variables that are used for the optimum nominal trajectory. Second, this correction scheme could be extended readily to optimum problems with more general types of end conditions than those considered in this report. Finally, a more general differential correction scheme is required for the optimum pursuit of a powered spacecraft, which would involve a statistical-control scheme for the probability law of a randomly moving point.

APPENDIX

VARIATIONAL PARAMETERS

For the calculation of variations of the optimum space trajectories, there is a general matrix introduced that relates the variations of the state and adjoint variables at time t to those at time t_0. This matrix, called the general transition matrix, requires the computation of the partial derivatives of the state and adjoint variables at two different times, i.e., t_0 and T, and relates their linear variations at these times, including the optimum changes of the thrusting program.

When the thrust is "off," the system of equations for the adjoint variables is "adjoint" to the system of equations for the variations of the state variables, which, in this case, is homogeneous, and the transition matrix of the state variables is used for the calculations of the adjoint variables during the coasting intervals of time, i.e., $t_i<t<t_{i+1}$. In this case, the transition matrix of the state variables $\hat{X}(t_{i+1}, t_i)$ is found from the corresponding Kepler problem, and it is expressed in closed form from the solution of this problem.

The variations of the state variables and the values of the adjoint variables for the coasting interval are given by [3].

$$\Delta\hat{\underline{x}}(t_{i+1}) = \hat{X}(t_{i+1}, t_i)\, \Delta\hat{\underline{x}}(t_i)$$
$$\hat{\underline{y}}(t_{i+1}) = \left[\hat{X}^T(t_{i+1}, t_i)\right]^{-1} \hat{\underline{y}}(t_i) \tag{45}$$

where

$$\hat{\underline{x}}(t)^T = (x_1, x_2, x_3, x_4, x_5, x_6)$$
$$\hat{\underline{y}}(t)^T = (y_1, y_2, y_3, y_4, y_5, y_6) \tag{46}$$

and

$$\hat{X}(t_{i+1}, t_i) = \frac{\partial\hat{\underline{x}}(t_{i+1})}{\partial\hat{\underline{x}}(t_i)} \tag{47}$$

The use of the conventional state variables $\hat{\underline{x}}(t)$, which are position and velocity vectors $\underline{R}$ and $\dot{\underline{R}}$ in cartesian coordinates, has the disadvantage that all of their elements have secular terms that vary rapidly with time. If, instead of the conventional state variables, other parameters are used as state variables, the resultant matrix might be simplified considerably. For example, consider the following parameters and their variations:

$\Delta\alpha_1$	Rotation of $\underline{R}$ about $\dot{\underline{R}}$
$\Delta\alpha_2$	Rotation of $\dot{\underline{R}}$ about $\underline{R}$
$\Delta\alpha_3$	Rotation of both $\underline{R}$ and $\dot{\underline{R}}$ about $\underline{H}$
$\Delta\alpha_4$	Change in cos $(\underline{R}, \dot{\underline{R}})$, keeping v and $\underline{R}$ constant
$\Delta\alpha_5$	Relative change in the semimajor axis $\Delta a/a$, keeping $\underline{R}$ and $\dot{\underline{R}}/v$ constant
$\Delta\alpha_6$	Relative change in the magnitude of the position vector $(\Delta r/r)$, keeping $\underline{R}/r$ and $\dot{\underline{R}}/v$ constant.

The transition matrix corresponding to the above parameters, i.e.

$$\Delta\underline{\alpha}(t) = \Psi(t, t_o)\, \Delta\underline{\alpha}(t_o) \tag{48}$$

$$
\Psi(t,t_o) = \begin{bmatrix} \frac{fv}{v_o} & \frac{-gv}{r_o} & 0 & 0 & 0 & 0 \\ \frac{-\dot{f}r}{v_o} & \frac{\dot{g}r}{r_o} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{\partial\alpha_3}{\partial\alpha_{40}} & \frac{\partial\alpha_3}{\partial\alpha_{50}} & \frac{\partial\alpha_3}{\partial\alpha_{60}} \\ 0 & 0 & 0 & \frac{\partial\alpha_4}{\partial\alpha_{40}} & \frac{\partial\alpha_4}{\partial\alpha_{50}} & \frac{\partial\alpha_4}{\partial\alpha_{60}} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{r_o v_o}{r^2} g & \frac{\partial\alpha_6}{\partial\alpha_{50}} & \frac{\partial\alpha_6}{\partial\alpha_{60}} \end{bmatrix} \tag{49}
$$

where some of the non-zero elements are listed as partials of the orbital parameters and are given by Ref. [4] as

$$
\frac{\partial\alpha_3}{\partial\alpha_{40}} = -\frac{r_o v_o}{r^2} g \left[\frac{r_o v_o}{h} \left(\frac{v_o g}{r_o} + f\alpha_{40} \right) + \frac{rh}{\mu g} \left(f - 1 \right) \left(\dot{g} - 1 \right) \right] \tag{50}
$$

$$
\frac{\partial\alpha_3}{\partial\alpha_{50}} = \frac{h}{r^2} \left[\frac{\mu}{v_o^2 r_o} g - \frac{3}{2}(t-t_o) - 3f\left(g - (t-t_o)\right) + (f-1)\frac{r_o}{v_o}\left(\alpha_{40} - \frac{rv}{r_o v_o} \alpha_4 \right) \right] \tag{51}
$$

$$
\frac{\partial\alpha_3}{\partial\alpha_{60}} = \frac{h}{r^2} \left[fg\left(1 - \frac{\mu}{r_o v_o^2} \right) - 2g + (f-1)^2 \frac{r_o}{v_o} \alpha_{40} \right] \tag{52}
$$

$$
\frac{\partial\alpha_4}{\partial\alpha_{40}} = \frac{r_o v_o}{rv} \left[\dot{g} - \frac{\mu\alpha_4}{r^2 v} \left(1 - \frac{r}{a} \right) g \right] \tag{53}
$$

$$
\begin{aligned}
\frac{\partial\alpha_4}{\partial\alpha_{50}} = \frac{\mu}{rv^2} \Bigg[& \left(1 - \frac{r}{a}\right) \left\{ \alpha_4 \left(f\frac{r_o}{v_o}\alpha_{40} + \dot{g} \right) - \frac{3v(t-t_o)}{2r}\left(1 - \alpha_4^2\right) \right\} - \frac{v}{v_o}\left(1 - \frac{r_o}{a}\right)\alpha_{40} \\
& + \frac{v}{r}\left\{ 1 - \left(1 - \frac{r}{a}\right)\alpha_4^2 \right\} \left\{ \frac{\mu}{v_o^2 r_o} g - \frac{r_o}{v_o}\left(\alpha_{40} - \frac{rv}{r_o v_o} \alpha_4 \right) \right\} \Bigg]
\end{aligned} \tag{54}
$$

$$\frac{\partial\alpha_4}{\partial\alpha_{60}} = \frac{\mu}{rv^2}\left[\frac{v}{v_o}\left(1-\frac{r_o}{a}\right)\alpha_{40} - \frac{v}{r}\left\{1-\alpha_4^{\,2}\left(1-\frac{r}{a}\right)\right\}\left\{g+\frac{r_o}{v_o}(f-1)\,\alpha_{40}\right\}\right.$$

$$\left. - \frac{\mu}{v_o^2 r_o}\left(1-\frac{r}{a}\right)\left\{\dot{g}+\frac{r_o}{r}\left(1-\frac{r}{a}\right)\right\}\right] \tag{55}$$

$$\frac{\partial\alpha_6}{\partial\alpha_{50}} = 1-\frac{r_o}{v_o}\,\dot{f}\,\alpha_{40} - \dot{g} + \frac{v}{r}\,\alpha_4\left[\frac{\mu}{v_o^{\,2} r_o}\,g - \frac{r_o}{v_o}\left(\alpha_{40}-\frac{rv}{r_o v_o}\alpha_4\right)-\frac{3}{2}(t-t_o)\right] \tag{56}$$

$$\frac{\partial\alpha_6}{\partial\alpha_{60}} = \frac{\mu}{v_o^{\,2} r_o}\left[\dot{g}+\frac{r_o}{r}\left(1-\frac{r}{a}\right)\right] - \frac{v\alpha_4}{r}\left[g+\frac{r_o}{v_o}(f-1)\,\alpha_{40}\right] \tag{57}$$

The transformation relating the variation of the conventional state variable $\Delta\hat{\underline{x}}^T = (\Delta\underline{R}, \Delta\dot{\underline{R}})$ to the variations of the above set of parameters $\Delta\underline{\alpha}^T = (\Delta\alpha_1, \Delta\alpha_2, \ldots \Delta\alpha_6)$ is given by

$$\Delta\hat{\underline{x}}(t) = P(t)\,\Delta\underline{\alpha}(t) \qquad \text{and} \qquad \Delta\underline{\alpha}(t) = P(t)^{-1}\,\Delta\hat{\underline{x}}(t) \tag{58}$$

where

$$P(t) = \begin{bmatrix} \frac{-\underline{H}}{v} & 0 & \frac{\underline{H}\times\underline{R}}{h} & 0 & 0 & \underline{R} \\ 0 & \frac{\underline{H}}{r} & \frac{\underline{H}\times\dot{\underline{R}}}{h} & \frac{-rv}{R^2}\,\underline{H}\times\dot{\underline{R}} & \frac{\mu}{2v^2a}\,\dot{\underline{R}} & \frac{-\mu}{rv^2}\,\dot{\underline{R}} \end{bmatrix} \tag{59}$$

and

$$P(t)^{-1^T} = \begin{bmatrix} \frac{-v\underline{H}}{h^2} & 0 & \frac{\underline{H}\times\underline{R}}{hr^2} & \frac{\underline{H}\times\underline{R}}{r^3v} & \frac{2a}{r^3}\,\underline{R} & \frac{\underline{R}}{r^2} \\ 0 & \frac{r\underline{H}}{h^2} & 0 & \frac{-\underline{H}\times\dot{\underline{R}}}{rv^3} & \frac{2a}{\mu}\,\dot{\underline{R}} & 0 \end{bmatrix} \tag{60}$$

The relationship between the transition matrix $\hat{X}(t, t_o)$ for the conventional state variables $\underline{\hat{x}}(t)$ and $\Psi(t, t_o)$ for the above set of parameters $\underline{\alpha}(t)$ is given by

$$\hat{X}(t, t_o) = P(t)\, \Psi(t, t_o)\, P(t_o)^{-1} \text{ and } \Psi(t, t_o) = P(t)^{-1} \hat{X}(t, t_o)\, P(t_o) \tag{61}$$

The scalar functions $f, g, \dot{f}$, and $\dot{g}$ are given by

(Elliptic)	(Hyperbolic)
$f = \frac{a}{r_o}(\cos\theta - 1) + 1$	$f = \frac{a}{r_o}(\cosh\theta - 1) + 1$
$g = (t-t_o) - \frac{\theta - \sin\theta}{n}$	$g = (t-t_o) - \frac{\sinh\theta - \theta}{n}$
$\dot{f} = \frac{-a^2 n}{r r_o}\sin\theta$	$\dot{f} = -\frac{\sqrt{-\mu a}}{r r_o}\sinh\theta$
$\dot{g} = \frac{a}{r}(\cos\theta - 1) + 1$	$\dot{g} = \frac{a}{r}(\cosh\theta - 1) + 1$

(62)

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